SOLVABLE GROUPS OF AUTOMORPHISMS OF COMPACT RIEMANN SURFACES

By

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A THESIS SUBMITTED

TO THE GAUHATI UNIVERSITY

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This is to certify that this thesis titled "Solvable groups of automorphisms of compact Riemann surfaces" is the outcome of study and investigations carried out by Ms. Gayatree Das and it has been done under my supervision and guidance.

This work or part thereof has not been submitted for any degree or diploma at this or any other university.

Ms. Gayatree Das fulfils the requirements of the regulations relating to the nature and prescribed period of research for the award of the degree of Doctor of Philosophy in this university.

KPatra

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SOLVABLE GROUPS OF AUTOMORPHISMS OF COMPACT RIEMANN SURFACES

A SYNOPSIS

A surface S is a connected Hausdorff space which is locally homeomorphic to the complex plane C(= $\mathbb{R}^2$). The surface S is called a ***Riemann surface*** :

(i) if there is a collection $\{U_i, f_i\}_{i \in \wedge}$ where, for the index set $\wedge$, $\{U_i\}$ is an open covering of S and f_i is a homeomorphism of U_i onto an open set in the complex plane C and

(ii) if, when $U_i \cap U_j \neq \phi$, then $f_j f_i^{-1}$ is a conformal sense preserving mapping of $f_i(U_i \cap U_j)$ onto $f_j(U_i \cap U_j)$; that is $w = f_j f_i^{-1}(z) = \Psi(z)$ is analytic function of z in $f_i(U_i \cap U_j)$. The condition (i) and (ii) are said to constitutes an analytic structure on the surface S, that is $A = \{U_i, f_i\}_{i \in \wedge}$ is an analytic structure on S. Thus a surface S with analytic structure A is a Riemann surface.

A compact Riemann surface is topologically equivalent to a sphere with a finite number of handles and this finite number is called the ***genus*** of the surface. It should be noted that there may be more than one Riemann surface of same genus depending on the nature of the analytic structure. The study of automorphisms of Riemann surfaces is the same as the study of birational self transformations of algebraic curves. The automorphisms of a compact Riemann surface of genus $g \geq 2$ is finite and its order $\leq 84(g - 1)$.

In the study of Riemann surface automorphism groups the theory of Fuchsian groups play a very important role. Since the early sixties of this century Fuchsian groups are extensively used in the study of the groups of automorphism (biholomorphic self transformations) of compact Riemann surfaces of genus $g \geq 2$. The lead was taken by Macbeath. He seems to be inspired by some works on the determination of groups of birational self transformations of algebraic curves by Hurwitz during the last decade of the last century. It may be noted that an algebraic curve is equivalent to a compact

Riemann surface.

In the mean time Fuchsian group theory can be recognised as a standard and very useful tool in dealing with problems of Riemann surface automorphism groups.

A Fuchsian group is abstractly defined as an infinite group generated by s-elements $x_1, \ldots, x_s$ of finite order and 2γ-elements $a_1, b_1, \ldots, a_\gamma, b_\gamma$ of infinite order fulfilling the defining relations :

$$x_1^{m_1} = \ldots = x_s^{m_s} = \prod_{i=1}^{s} x_i \prod_{i=1}^{\gamma} [a_i, b_i] = 1$$

with the condition,

$$\delta(\Gamma) = 2\gamma - 2 + \sum_{i=1}^{s}\left(1 - \frac{1}{m_i}\right) > 0.$$

Such a Fuchsian group is usually denoted $(\gamma; m_1, \ldots, m_s)$. The non-negative integer γ is the ***genus*** of the Fuchsian group and the integers $m_i \geq 2$ are the ***periods*** of Γ. When $s = 3$, $\gamma = 0$ and $s = 4$, $\gamma = 0$ then Γ is called ***triangle*** and ***quadruple*** group respectively. A Fuchsian group K without any finite order elements is called a ***surface group*** and if g is the genus of K then $\delta(K) = 2(g - 1)$.

If Γ_1 is a subgroup of Γ of finite index then

$$[\Gamma : \Gamma_1] = \frac{\delta(\Gamma_1)}{\delta(\Gamma)}.$$

If $\Gamma = (\gamma; m_1, \ldots, m_s)$ and K is a normal surface subgroup of Γ of genus g, then the quotient group Γ/K is called a ***smooth-quotient*** of genus g. A homomorphism ϕ from a Fuchsian group Γ onto a finite group G is a smooth homomorphism if the kernel is a surface group.

The study of the quotients of Fuchsian group is important because if a finite group G is a smooth quotient of genus $g \geq 2$ of some Fuchsian group then G acts as an automorphism group of compact Riemann surface of genus g. Also the finite quotients of Fuchsian triangle groups give two generator finite groups, that is groups with minimal sets of generators in case of non-cyclic groups.

The technique that we have used in our thesis leads us to study the famous minimum genus problem in the theory of Riemann surface automorphism groups in relation to a particular Zs-metacyclic group of order $4p^2$. This minimum genus problem was studied by Harvey considering the cases of cyclic groups and by Maclachlan considering the cases of non-cyclic abelian groups.

We now give below a chapterwise brief discussion of the thesis.

In chapter 1, we discuss Fuchsian groups, some important and significant results and establish the link between Fuchsian groups and Riemann surface automorphism groups. This chapter also contains some early important result which are necessary in the field of our study.

In chapter 2, we find some infinite families of two generator finite solvable groups with short derived series. The main results are contained in the following theorems.

Theorem 2.1.1 [50] : Let $\Gamma = (\ell, m, n)$ be a Fuchsian group where ℓ, m, n are positive integers ≥ 2 such that $(\ell, m) = d > 1$ and $(\ell, n) = (m, n) = 1$. ℓ, m, n donot simultaneously assume the values $\ell = m = 2$ or $\ell = m = 3$, $n = 2$. Then for each positive integer $k \geq 1$, $\Gamma = (\ell, m, n)$ has a solvable smooth quotient G_s of derived length 3 and of order

$$(kt)^{2\gamma''} dn^{d-1} a^{n^{d-1}-1} b^{n^{d-1}-1}.$$

Theorem 2.1.2 : Let $\Gamma = (\ell, m, n)$ be a Fuchsian group where ℓ, m, n are positive integers ≥ 2 such that $(\ell, m) = d_1 > 1$, $(\ell, n) = d_2 > 1$, $(m, n) = 1$ and $(d_1, d_2) = 1$ (as $(m, n) = 1$). Then for each positive integer $k > 1$, Γ admits smooth quotients of order :

$$d_1 d_2 k^{2\gamma'} t^{2\gamma_k} a^{A+2\gamma_k-1} b^{A/h_1+2\gamma_k-1} c^{A/h_2+2\gamma_k-1} h_1^{d_2-A/h_1-2\gamma_k} h_2^{d_1-A/h_2-2\gamma_k}, k > 1$$

and of genus :

$$\frac{1}{2h_1^{A/h_1+2\gamma_k-1}h_2^{A/h_2+2\gamma_k-1}}\left(a^{A+2\gamma_k-1}b^{A/h_1+2\gamma_k-1}c^{A/h_2+2\gamma_k-1}t^{2\gamma_k}\right)\Bigg[2\gamma_k-2$$

$$+A\left(1+\frac{1}{h_1}+\frac{1}{h_2}-\frac{1}{a}-\frac{1}{b}-\frac{1}{c}\right)\Bigg]+1$$

where $A = h_1^{d_2-1}h_2^{d_1-1}k^{2\gamma'}$.

Theorem 2.1.3 : Let $\Gamma = (\ell, m, n)$ be a Fuchsian group where ℓ, m, n are positive integers ≥ 2 such that $(\ell, m, n) = r \geq 1$ and $(\ell, m) = rd_1$, $(\ell, n) = rd_2$, $(m, n) = rd_3$, for some non-negative integers d_1, d_2 and d_3, where d_1, d_2, d_3 are pairwise prime to each other. Then Γ admits a metabelian smooth quotients of order :

$$r^2d_1^2d_2d_3k^{2\gamma'}a^{rd_1d_3+2\gamma'-1}b^{rd_1d_2+2\gamma'-1}c^{rd_1^2+2\gamma'-1}$$

and of genus :

$$\frac{1}{2}\left(a^{rd_1d_3+2\gamma'-1}b^{rd_1d_2+2\gamma'-1}c^{rd_1^2+2\gamma'-1}k^{2\gamma'}\right)\Bigg[2\gamma'-2$$

$$+rd_1\left(d_1+d_2+d_3-\frac{d_3}{a}-\frac{d_2}{b}-\frac{d_1}{c}\right)\Bigg]+1.$$

In chapter 3, we study a class Zs-metacyclic group M of order $4p^2$ which are two generator groups, generated by a, b and having the following presentation

$$\left\langle a,b : a^{p^2} = b^4 = 1, b^{-1}ab = a^{-1}, \text{where p is an odd prime}\right\rangle.$$

We first obtain a set of necessary and sufficient conditions on the periods and genus of Γ. Then we use these conditions to obtain the minimum genus of a compact Riemann surface that has G as an automorphism group. In this chapter the main theorem determines the minimum genus of a surface admitting the given Zs-metacyclic group as an automorphism group. The theorem is given below :

Theorem 3.2.1 : Let M be a Zs-metacyclic group of order $4p^2$ where p is an odd prime, with presentation :

$$\left\langle a, b : a^{p^2} = b^4 = 1, b^{-1}ab = a^{-1} \right\rangle.$$

Let M act as an automorphism group of some compact Riemann surface of genus $g \geq 2$. Then the minimum value of g and the corresponding signature of the Fuchsian group Γ of which M is a smooth quotient is :

$$g = p^2 - 2p + 1;\ (4, 4, p).$$

In the last chapter i.e. in chapter 4 our aim is to find finite solvable admissible quadruple groups and in the last section of this chapter we find the solvable extension of admissible quadruple groups and corresponding genus of surface of which these groups act as a group of automorphism of compact Riemann surfaces. The main theorems are given below :

Theorem 4.1.1 : Let $\Gamma = (\ell, m, n, \mu)$ be a Fuchsian group where ℓ, m, n, μ are positive integers greater than or equal to 2 such that, $(\ell, m) = d_1 > 1$, $(\ell, n) = (\ell, \mu) = (m, n) = (m, \mu) = (n, \mu) = 1$. Then Γ admits a solvable smooth quotient of derived length 3 and of order $k^{2\gamma''} d_1 (n\mu)^{d_1-1} (ab)^{(n\mu)^{d_1-1}-1+2\gamma''}$ and genus

$$\frac{1}{2} k^{2\gamma''} (ab)^{(n\mu)^{d_1-1}-2+2\gamma''} \left[2ab(\gamma''-1) + (n\mu)^{d_1-1}(2ab-b-a) \right] + 1,\ k \geq 1$$

where $$\gamma'' = \frac{n^{d_1-1}\mu^{d_1-1}}{2} \left[2d_1 - 2 - \frac{d_1}{n} - \frac{d_1}{\mu} \right] + 1.$$

Theorem 4.1.2 : Let $\Gamma = (\ell, m, n, \mu)$ be a Fuchsian group where ℓ, m, n, μ are positive integers greater than or equal to 2 such that $(\ell, m) = d_1 > 1$, $(\ell, n) = d_2 > 1$ and $(\ell, \mu) = (m, n) = (m, \mu) = (n, \mu) = 1$; d_1, d_2 are prime to each other. Then Γ admits a finite smooth quotient of order $d_1 d_2 AB$ where

$$A = h_1^{d_2(d_1+1)-2} h_2^{d_1(d_2+1)-2} k^{2\gamma'}; k, h_1, h_2 \geq 1$$

$$B = a^{A-1}\left(\frac{b}{h_1}\right)^{A/h_1 - 1}\left(\frac{c}{h_2}\right)^{A/h_2 - 1}\left(\frac{\mu}{h_3}\right)^{A/h_3 - 1}(t\ell')^{2\gamma_k},$$

where $\ell' = \dfrac{abc\mu}{h_1h_2h_3}$; and of genus :

$$\frac{B}{2}\left[2\gamma_k - 2 + A\left(1 + \frac{1}{h_1} + \frac{1}{h_2} + \frac{1}{h_3} - \frac{1}{a} - \frac{1}{b} - \frac{1}{c} - \frac{1}{\mu}\right)\right] + 1.$$

Theorem 4.1.3 : Let $\Gamma = (\ell, m, n, \mu)$ be a Fuchsian group where ℓ, m, n, μ are positive integers greater than or equal to 2 such that $(\ell, m) = d_1 > 1$, $(\ell, n) = d_2 > 1$, $(\ell, \mu) = d_3 > 1$ and $(m, n) = (m, \mu) = (n, \mu) = 1$, where d_1, d_2, d_3 are prime to each other. Then Γ admits a finite smooth quotient of Γ of genus

$$\frac{B}{2}\left[2\gamma_k - 2 + A\left(1 + \frac{1}{h_1} + \frac{1}{h_2} + \frac{1}{h_3} - \frac{1}{a} - \frac{1}{b} - \frac{1}{c} - \frac{1}{r}\right)\right] + 1$$

and the order is $d_1d_2d_3AB$; where

$$A = h_1^{d_2d_3-1}h_2^{d_1d_3-1}h_3^{d_1d_2-1}k^{2\gamma'}$$

$$B = a^{A-1}\left(\frac{b}{h_1}\right)^{A/h_1 - 1}\left(\frac{c}{h_2}\right)^{A/h_2 - 1}\left(\frac{r}{h_3}\right)^{A/h_3 - 1}(t\ell')^{2\gamma_k}$$

$$\ell' = \frac{abcr}{h_1h_2h_3}$$

and $$\gamma_k = \frac{A}{2}\left[2\gamma' - 2 + d_2d_3\left(1 - \frac{1}{b}\right) + d_1d_3\left(1 - \frac{1}{c}\right) + d_1d_2\left(1 - \frac{1}{r}\right)\right.$$

$$\left. - \frac{1}{h_1} - \frac{1}{h_2} - \frac{1}{h_3} + \frac{1}{b} + \frac{1}{c} + \frac{1}{r}\right] + 1.$$

Theorem 4.1.4 : Let $\Gamma = (\ell, m, n, \mu)$ be a Fuchsian group where ℓ, m, n, μ are positive integers ≥ 2 such that $(\ell, m) = rd_1$, $(\ell, n) = rd_2$, $(m, n) = rd_3$, $(\ell, m, n) = r \geq 1$ and $(\ell, \mu) = (m, \mu) = (n, \mu) = 1$, where d_1, d_2, d_3 are pairwise prime to each other. Then Γ admits a metabelian smooth quotient of order $Ar^2d_1{}^2d_2d_3$ and of genus :

$$\frac{A}{2}\left[2r^2d_1^2d_2d_3 - \frac{rd_1d_3}{a} - \frac{rd_1d_2}{b} - \frac{rd_1^2}{c} - \frac{r^2d_1^2d_2d_3}{\mu}\right] + 1$$

where $A = a^{rd_1d_3+2\gamma'-1}b^{rd_1d_2+2\gamma'-1}c^{rd_1^2+2\gamma'-1}\mu^{r^2d_1^2d_2d_3+2\gamma'-1}k^{2\gamma'}, k \geq 1$

and $\gamma' = \frac{1}{2}r[rd_1^2d_2d_3 - d_1d_3 - d_1d_2 - d_1^2] + 1.$

Theorem 4.1.5 : Let $\Gamma = (\ell, m, n, \mu)$ be a Fuchsian group where ℓ, m, n, μ are positive integers greater than or equal to 2 such that $(\ell, m) = d_1 > 1$, $(\ell, n) = d_2 > 1$, $(\ell, \mu) = d_3 > 1$, $(m, n) = d_4 > 1$ and $(m, \mu) = (n, \mu) = 1$, where d_1, d_2, d_3, d_4 are pairwise prime to each other. Then Γ admits a metabelian smooth quotient of order, $Ad_1d_2{}^2d_3d_4$ and of genus :

$$\frac{A}{2}\left[2d_1d_2^2d_3d_4 - \frac{d_2d_4}{a} - \frac{d_2^2d_3}{b} - \frac{d_1d_2d_3}{c} - \frac{d_1d_2^2d_4}{d}\right] + 1$$

where $A = a^{d_2d_4+2\gamma'-1}b^{d_2^2d_3+2\gamma'-1}c^{d_1d_2d_3+2\gamma'-1}d^{d_1d_2^2d_4+2\gamma'-1}k^{2\gamma'}, k \geq 1$

and $\gamma' = \frac{1}{2}[2d_1d_2^2d_3d_4 - d_2d_4 - d_2^2d_3 - d_1d_2d_3 - d_1d_2^2d_4] + 1.$

Theorem 4.3.1 : Let G be an admissible quadruple group of type (ℓ, m, n, ξ) generated by u, v, w such that $p_1 < \dots < p_k$ and p_i not divide m_{i-1}, n_{i-1} and ξ_{i-1} where m_i, n_i and ξ_i are defined as follows :

(i) $m_0 = m$, $m_1 = [m_0, p_1]$, $m_2 = [m_1, p_2]$, ...

(ii) $n_0 = n$, $n_1 = [n_0, p_1]$, $n_2 = [n_1, p_2]$, ...

(iii) $\xi_0 = \xi$, $\xi_1 = [\xi_0, p_1]$, $\xi_2 = [\xi_1, p_2]$, ...

Then p_i-th extension G_i of G is an automorphism group of a compact Riemann surface of genus g_i, $1 \leq i \leq k$ where

$$g_i = 1 + \frac{1}{2}(p_1 \ldots p_i)|G|\left\{2 - \frac{1}{\ell} - \frac{1}{m_i} - \frac{1}{n_i} - \frac{1}{\xi_i}\right\}.$$

At the end of the thesis a fairly exhaustive bibliography is added.

* * *

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Ms. Gayatree Das
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Chapter 1 : Introduction

CHAPTER - 1

INTRODUCTION

1.1 Importance of our study

The theory of Riemann surface was developed during the second half of the nineteenth century when Riemann introduced it in his doctoral dissertation in 1851 as essentially topological aids to our understanding of many-valued functions. Though the results stated by Riemann was powerful but the proofs were not completely rigorous as the necessary analytic techniques had not been fully developed. This omission was rectified mainly by weierstrass and by the early twentieth century Riemann's theory had been placed on a sound basis. One of the most interesting result observed by Riemann was that the algebraic function fields in one variable over C are nothing but fields of meromorphic functions on compact Riemann surfaces. Hence groups of birational automorphisms of complex algebraic curves are the same as automorphism groups of compact Riemann surfaces. This new point of view drew the attention of the mathematical world and propelled it to concentrate on complex function theory, mainly on the study of meromorphic function of a complex variable. A related classical topic consists their groups of birational automorphisms which was initiated also in the last century.

Complex algebraic curves and their groups of birational transformations constituted a glamourous topic of research during the last decade of the last century. Surprisingly this topic continues to attract the attention of mathematicians even after a century. At the present time there is a great revival of interest in these topics because of their applications to so many areas of mathematical research from group theory and number theory to topology and differential equation.

1.2 A survey of some significant early results

We now mention some significant results in this field which have been proved in the last century.

The first significant result in this field was published by H.A. Schwarz in 1879 [121] in which he proved the finiteness of the group of automorphisms of complex algebraic curves C of genus $g \geq 2$.

About fifteen years later, A. Hurwitz [1893] using his famous ramification formula obtained an upper bound on the order of A(C) of automorphisms of C and showed that the order of G, i.e. $| G | \leq 84(g - 1)$ where G is the group of automorphisms of curves C of genus $g \geq 2$. It has been showed by Klein, Gordan, Wiman [84] [67] [134] that this bound, known as Hurwitz bound, is not attained for genus $g = 2, 4, 5, 6$ and there is only one curve of genus 3 with $168 = 84(g - 1)$ automorphisms. About the same time Wiman improved this bound for a cyclic group and showed that the maximum possible order for a group of birational automorphisms is $2(2g + 1)$ and this bound is attained infinitely many times. The general strategy for a better understanding of automorphism groups of a compact Riemann surface S of genus $g \geq 2$ is explained by Macbeath [96] which was paraphrased by Accola [4].

The general problem on the theory of Riemann surface that can be analyzed is as follows : given a class $\mathscr{C}$ of finite groups and a class $\mathscr{K}$ of compact Riemann surfaces of algebraic genuses $g \geq 2$, under what conditions a surface S in $\mathscr{K}$ and a group G in $\mathscr{C}$ exist, so that G acts as a group of automorphisms on S?

Some consequences of the above general problem is to determine the minimum genus of surfaces admitting an automorphism group of a given order and to determine the maximum order of an automorphism group of a surface of given genus.

Another remarkable result [Burnside 1955] is that for any finite group G, there exists a compact Riemann surface S on which G acts as a group of automorphisms of S. The same group may be representable as an automorphism group of compact Riemann surfaces of different genera.

In the early sixties A. M. Macbeath [95] gave a series of lectures in a summer school at Queen's college, Dundee where he dealt with a number of interesting problems and some of their solutions. The results of Macbeath was based on the theory of Fuchsian groups which reduced the general problem on the theory of Riemann surface to a purely algebraic one. The main tool of Macbeath's study was based on a result of Poincare [120] which stated that each compact Riemann surface S of (algebraic) genus $g \geq 2$ can be represented as an orbit space D/Γ of the upper half complex plane D, that is $D = \{z \in C : \text{Im}\, z \geq 0\}$ and if D/Γ is compact then S is also compact. Here D is equipped with the conformal structure induced by the group Ω of Möbius transformations, and the acting group Γ is a Fuchsian group which is a discrete subgroup of Ω. The group Γ can be choosen to be a ***surface-group*** that is a group with no element of finite order. With this representation in hand Macbeath proved that a finite group G is a group of automorphisms of a compact Riemann surface S if and only if $G \cong \wedge/\Gamma$ for some other Fuchsian group $\wedge$ [95]. The quotient group $\wedge/\Gamma$ is known as ***smooth quotient*** of $\wedge$.

In this new approach of Macbeath the theory of Fuchsian groups seems to be the most powerful tool in order to investigate the structure of automorphism groups of complex algebraic curves.

In the next section we discuss in brief the theory of Fuchsian groups and Riemann surface automorphism groups. Before going to the next section we first introduce the concept of a surface.

A surface S is a Hausdorff topological space which is locally homeomorphic to the complex plane C.

A Riemann surface S is a surface equipped with an analytic structure A, that is a family A of pairs (v_i, f_i) satisfying the conditions :

(i) for the index set I, $\{v_i\}_{i \in I}$ is an open covering of S and f_i is a homeomorphism of v_i onto an open disc in the complex plane C.

(ii) when $v_i \cap v_j \neq \phi$ then $f_j.f_i^{-1}$ is an analytic function of $f_i(v_i \cap v_j)$ onto $f_j(v_i \cap v_j)$.

A continuous mapping f from S to itself is called ***holomorphic*** if for any two pairs $(v_1, f_1), (v_2, f_2) \in A$, the mapping $f_2.f.f_1^{-1}$ is holomorphic in the domain of definition $f_1(v_1 \cap f^{-1}(v_2))$. If f is one-one and onto it is called ***biholomorphic self transformation*** or automorphism of S. The set of all automorphisms of S forms a group A(S) and is called the ***group of automorphisms*** or the ***automorphism group*** of S.

A compact Riemann surface is topologically equivalent to a sphere with a finite number of handles and this finite number is called the ***genus*** of the surface. It would be noted that there may be more than one Riemann surface of same genus depending on the nature of the analytic structure.

Macbeath established a link between the Riemann surface automorphism groups and the Fuchsian groups and proved that a finite group G is an automorphism group of compact Riemann surface S of genus $g \geq 2$ if and only if G is isomorphic to a factor group Γ/K where Γ is a Fuchsian group with compact ***orbit space*** and K is a Fuchsian surface group with ***orbit genus*** g. The quotient Γ/K is called a smooth quotient and the corresponding homomorphism is called a smooth homomorphism. Our technique depends heavily on the result of Macbeath.

1.3 Fuchsian groups and automorphisms of compact Riemann surfaces

If X is the Riemann sphere i.e. the set of all complex numbers Z together with ∞ then the infinite set of transformations $z' = \dfrac{az+b}{cz+d}$ of the Riemann sphere X, where a, b, c, d are real numbers and $ad - bc \neq 0$ constitutes a general linear group and is denoted by $GL(2, \mathbb{R})$. The group consisting of all 2×2 real matrices with determinant unity is the ***special linear group*** and is denoted by $SL(2, \mathbb{R})$. $SL(2, \mathbb{R})$ is a subgroup of GL(2, $\mathbb{R}$). The ***projective special linear group*** PSL(2, $\mathbb{R}$) is the central quotient of SL(2, $\mathbb{R}$) i.e.,

$$\mathrm{PSL}(2,\mathbb{R}) \cong \mathrm{SL}(2,\mathbb{R})/Z[\mathrm{SL}(2,\mathbb{R})]$$

where Z[SL(2,$\mathbb{R}$)] is the centre of SL(2,$\mathbb{R}$).

It can be seen that PSL(2,$\mathbb{R}$) maps :

(i) The open upper half plane D to itself.

(ii) The open lower half plane D' to itself.

(iii) $\mathbb{R} \cup \{\infty\}$ to itself.

Further PSL(2,$\mathbb{R}$) preserves,

(iv) the angle between two curves and

(v) the family of all circles and straight lines orthogonal to the real axis.

The elements of PSL(2, $\mathbb{R}$) can be classified as parabolic, hyperbolic and elliptic according to the trace, i.e. an element is parabolic if $|a+d| = 2$, hyperbolic if $|a + d| > 2$ and elliptic if $|a + d| < 2$. It can be seen that all elements of finite order are elliptic.

It is to be noted that any subgroup of PSL(2, $\mathbb{R}$) acts as the group of all conformal homeomorphisms of D when D is endowed with the hyperbolic non-Euclidean (N.E.) metric given by $ds^2 = \dfrac{dx^2 + dy^2}{y^2}$, $z = x + iy \in C$. D becomes a model of hyperbolic plane, and PSL(2, $\mathbb{R}$) acts as a group of hyperbolic isometries. PSL(2, $\mathbb{R}$) besides being a group, is also a topological space in that the transformtion $z \to \dfrac{az+b}{cz+d}$ can be identified with the point $(a, b, c, d) \in \mathbb{R}^4$. More precisely, as a topological space, SL(2, $\mathbb{R}$) can be identified with the subset of $\mathbb{R}^4$.

Let $X = \{(a, b, c, d) \in \mathbb{R}^4 : ad - bc = 1\}$ and if we define $\delta(a, b, c, d) = (-a, -b, -c, -d)$ then $\delta : X \to X$ is a homeomorphism and δ together with the identity forms a cyclic group of order 2 acting on X. PSL(2, $\mathbb{R}$) can be topologised as the quotient space. It can also be shown that the group multiplication and taking of inverses are continuous with this topology so that PSL(2,IR) is a topological group.

A subgroup Γ of PSL(2,IR) is said to be ***discrete*** if for all $g \in \Gamma$, there exists a neighbourhood U_g such that $U_g \cap \Gamma = \{g\}$. If Γ is a discrete subgroup of PSL(2,IR) then it is called a ***Fuchsian group***. The connection between the group PSL(2,IR) and hyperbolic geometry was discovered by Poincare [120] and published in 1882.

On the upper half plane D, a Non-Euclidean (N. E.) geometry can be established by defining the N. E. length 'ds' of an arc element by .

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

The ***N. E. length*** $\ell(c)$ of a piecewise differentiable curve C in D is defined by

$$\ell(c) = \int \frac{|dz|}{y}, z = x + iy,$$

and the ***N. E. measure*** $\mu(E)$ of a measurable set E in D is defined by

$$\mu(E) = \iint_E \frac{dxdy}{y^2}.$$

The hyperbolic length and area are invariant under transformations of PSL(2, IR) and hence of Γ.

The angles in this geometry are the normal Euclidean angles. The circles and straight lines orthogonal to the real axis are called the N. E. lines. The N. E. line joining two points of D is the (Euclidean) circle passing through the points, which has its centre on the real axis. This definition includes the Euclidean lines which are vertical to the real axis. The ***N. E. distance*** d(p, q) between two points p and q in D is defined to be the N. E. length of the N. E. line segment joining p and q. The topology induced by the N. E. metric however coincides with the topology obtained by regarding D as a subspace of IR^2. Let Γ be an arbitrary Fuchsian group and let $p \in D$ be not fixed by any element of $\Gamma \backslash \{I\}$, then we define the ***Dirichlet region*** for Γ at p to be the set,

$$F = \{z \in D \mid d(z, p) \leq d(z, v(p)) \text{ for } v \in \Gamma\},$$

d being the hyperbolic (N. E.) metric on D. A specially constructed fundamental region is a Dirichlet region consisting of a finite number of N. E. edges forming a polygon (N. E.). It is to be noted that,

$$D = \cup\{v(F) \mid v \in \Gamma\},$$

the different images v(F) having mutually disjoint interiors.

Fuchsian groups are discrete groups of hyperbolic isometries and their quotient spaces are also Riemann surfaces. The distinct images of a point z in D under Γ give rise to an ***orbit*** known as Γ orbit and the orbits with the topology given by identification, form the orbit space denoted by D/Γ. If the orbit space D/Γ is compact that is if D/Γ has a finite subcover then the Fuchsian group Γ is said to be ***co-compact***. Now by a result of Massey(1967) [104] the orbit space D/Γ is homeomorphic to a surface S_γ, formed by attaching γ handles to a sphere, for some unique integer $\gamma \geq 0$, which is called the ***genus*** of the surface. The genus γ of the orbit space is called the ***genus*** of the Fuchsian group Γ.

Given an element z in D, the set of elements of Γ with z as a fixed point (i.e. the set of transformations which fix z) is known as the stabilizer of z in Γ. The stabilizer of a point of D in Γ is always the identity or a finite cyclic group.

The only non-trivial finite cyclic subgroups of PSL(2, $\mathbb{R}$) are those generated by elliptic elements and each elliptic element has a unique fixed point in D. The same is true for Γ also. Hence every element of finite order in Γ belongs to a maximal finite cyclic subgroup of Γ. There are infinitely many of these maximal finite cyclic subgroups if there is one, but they fall into a finite number of conjugacy classes. The orders of these maximal finite cyclic subgroups are called the ***periods*** of Γ. The ***multiplicity*** of a period is the number of distinct conjugacy classes of maximal finite cyclic subgroups with that period for their order.

By a theorem of Siegel [123], a Fuchsian group Γ with compact orbit space has only a finite number of periods. The r-tuple $\{m_1, m_2, \ldots, m_r\}$ consisting of the periods of Γ in some order, but each repeated according to its multiplicity is called a ***period partition*** of Γ. Orbit genus and period partition of a co-compact Fuchsian group are invariant under group isomorphisms. The algebraic structure of Γ is completely determined when the period partition and the genus of the orbit space are known. If Γ is a co-compact Fuchsian group $(\gamma; m_1, \ldots, m_r)$ and F is a fundamental region for Γ then its hyperbolic area (N. E. measure) μ(F) is finite, bounded below by $\pi/21$ and is given by,

$$\mu(F) = 2\pi\left\{2(\gamma-1)+\sum_{i=1}^{r}\left(1-\frac{1}{m_i}\right)\right\}$$

which is always positive.

i.e. $$\delta(F) = 2(\gamma-1)+\sum_{i=1}^{r}\left(1-\frac{1}{m_i}\right) > 0\cdot$$

For a surface group of genus g the hyperbolic measure μ(F) = 4π(g - 1). It is to be noted that the N. E. measure μ(F) of a fundamental region F of Γ depends only on Γ. In all cases of Fuchsian groups with positive genus the measure of their fundamental regions ≥ π.

Any subgroup of finite index of a Fuchsian group is Fuchsian [73], [95] and if Γ_1 is a subgroup of finite index of a Fuchsian group Γ, then

$$[\Gamma:\Gamma_1] = \frac{\mu(F_1)}{\mu(F)}. \qquad \ldots\ldots(1.3.1)$$

This is a form of the famous Riemann Hurwitz formula (R. H. formula).

A Fuchsian group Γ with genus γ and a period partition $\{m_1, \ldots, m_r\}$ is generated by

$x_1, \ldots, x_r$ elements of finite order and $a_1, b_1, \ldots, a_\gamma, b_\gamma$ elements of infinite order and the generators satisfy :

$$x_1^{m_1} = \ldots = x_r^{m_r} = \prod_{i=1}^{r} x_i \prod_{i=1}^{\gamma} [a_i, b_i] = 1 \qquad \ldots\ldots(1.3.2)$$

where $[a_i, b_i] = a_i^{-1} b_i^{-1} a_i b_i$

and $$\delta(\Gamma) = 2(\gamma - 1) + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i} \right) > 0 . \qquad \ldots\ldots.(1.3.3)$$

Associated with a Fuchsian group is its signature $(\gamma; m_1, \ldots, m_r)$. If $\gamma = 0$ then we will denote the signature of Γ by $(m_1, \ldots, m_r)$ and if in addition $r = 3$ then we call Γ a ***triangle-group*** and if $r = 4$ then Γ is called a ***quadruple group***. It is better to mention that a Fuchsian group $\Gamma = (m_1, \ldots, m_s)$ with genus 0 and s periods can in fact be generated by $s - 1$ elements having defining relations :

$$x_1^{m_1} = x_2^{m_2} = \ldots = x_{s-1}^{m_{s-1}} = \left(x_1 x_2 \ldots x_{s-1} \right)^{m_s} = 1 . \qquad \ldots\ldots(1.3.4)$$

This will be obtained from the fact that the generator x_s can be eliminated by the last relation in the set of defining relations for (1.3.2).

Thus a triangle and quadruple groups can be generated by two elements and three elements respectively.

If the Fuchsian group Γ of genus γ has no period, we write $\Gamma = (\gamma; -)$, and then it is called a ***surface group*** that is a Fuchsian group without elements of finite order except the identity. It may be noted that any finite order element in Γ is conjugate to some of the finite order generators.

The numbers $\gamma, m_1, \ldots, m_s$ occuring in the signature of a Fuchsian group Γ must obey the inequality (1.3.3) and in view of this certain values of the periods m_i and genus γ are untenable for a Fuchsian group with compact orbit space. These values are the following :

(a) (m, m); the cyclic group Z_m.

(b) (2, 2, n); the dihedral group D_n.

(c) (2, 3, 3); the tetrahedral group.

(d) (2, 3, 4); the octahedral group.

(e) (2, 3, 5); the Icosahedral group.

The space groups of 2-dimensional crystallography namely,

(f) (2, 2, 2, 2)

(g) (3, 3, 3)

(h) (2, 3, 6)

(i) (2, 4, 4)

and (j) the doubly periodic group with orbit genus 1 and no period.

Moreover, the existence theorem according to Siegel ensures that for every set of tenable values of γ, m_1, m_2, ..., m_s there exists a Fuchsian group with signature (γ; m_1, ..., m_s), if and only if

$$\delta(\Gamma) = 2(\gamma - 1) + \sum_{i=1}^{s}\left(1 - \frac{1}{m_i}\right) > 0.$$

From the existence theorem it also follows that the orbit genus of a surface group with compact orbit space cannot be less than 2.

1.4 Riemann surface automorphism groups

This section is devoted towards gathering some useful information about Riemann surface automorphism groups and their relation to quotient groups of Fuchsian groups.

In section 1.2 we defined a surface and a Riemann surface. The universal covering space $\overline{S}$ of a Riemann surface is itself a Riemann surface.

The group of covering transformations of the universal covering space $\overline{S}$ of S is isomorphic to the fundamental group of S and is called the ***Poincare group* of S**. If K is the Poincare group of S, then the orbit space $\overline{S}/K$ is also a Riemann surface.

We now state some fundamental results about Riemann surfaces.

Theorem 1.4.1 : The Riemann surface S is identifiable with the orbit-space $\bar{S}/K$ by a structure preserving homeomorphisms where $\bar{S}$ is the universal covering space of S and K is the Poincare group of S.

Theorem 1.4.2 : If S is a Riemann surface with Poincare group K, then $A(S) \cong N(K)/K$ where N(K) is the normalizer of K in PSL(2, R) and A(S) is the group of automorphisms of S.

Theorem 1.4.3 : If S is a compact Riemann surface of genus $g \geq 2$, then A(S) is finite.

Proof : We know that there exists a surface group K such that S is conformally equivalent to D/K. Then by theorem 1.4.2,

$$A(S) \cong N(K)/K \,.$$

Now, since K is an orbit-compact surface group with orbit-genus at least 2, it is not cyclic and hence N(K) is a Fuchsian group containing an orbit-compact subgroup. Hence it has a compact orbit-space itself and hence a fundamental region of finite measure. In fact $\mu(F_N) \leq \mu(F)$ where F_N and F are fundamental regions for N(K) and K respectively. Now $[N(K):K] = \frac{\mu(F)}{\mu(F_N)}$ is finite, which proves that $A(S) \cong N(K)/K$ is finite.

We now prove a theorem which will be used as a basic tool in the development of the later chapters.

Theorem 1.4.4 : A finite group G acts as a group of automorphisms of a compact Riemann surface S of genus $g \geq 2$, if and only if G is isomorphic to a factor group Γ/K, where Γ is a Fuchsian group with compact orbit-space and K is a surface group with genus g.

Proof : Since $g \geq 2$, we have $S = D/K$. Then by theorem 1.4.2, the group G corresponds to some Fuchsian group Γ lying between K and N(K) with $G \cong \Gamma/K$.

Conversely, let $G \cong \Gamma/K$. Then K is a normal subgroup of Γ of finite index. Now $\Gamma \subseteq N(K)$ and N(K) maps K-orbits to K-orbits. Therefore Γ/K acts as a group of automorphisms of D/K.

The following corollary is obtained from the above theorem.

Corollary 1.4.1 : A finite group G acts as a group of automorphism of a compact Riemann surface of genus $g \geq 2$ if and only if there exists a Fuchsian group Γ with compact orbit space and a smooth epimorphism $\phi : \Gamma \rightarrow G$ such that $\ker\phi$ is a surface group of genus g.

From the above theorem and the relation (1.3.1) we get, $|G| = \left|\Gamma/K\right| = \dfrac{\mu(F_K)}{\mu(F_\Gamma)}$ where F_K and F_Γ are fundamental regions for K and Γ respectively. If $\Gamma = (\gamma; m_1, \ldots, m_s)$ then by using the relations :

$$\mu(F) = 2\pi\left\{2\gamma - 2 + \sum_{i=1}^{s}\left(1 - \frac{1}{m_i}\right)\right\}$$

and $\mu(F_K) = 4\pi(g - 1)$,

we have $$|G| = \frac{4\pi(g-1)}{2\pi\left\{2\gamma - 2 + \sum_{i=1}^{s}\left(1 - \frac{1}{m_i}\right)\right\}}.$$

i.e., $$\frac{2(g-1)}{|G|} = 2\gamma - 2 + \sum_{i=1}^{s}\left(1 - \frac{1}{m_i}\right)$$

which is the Riemann Hurwitz formula. By the relation (1.3.3) the expression :

$2(\gamma - 1) + \sum_{i=1}^{s}\left(1 - \frac{1}{m_i}\right)$ is strictly positive. Therefore for a fixed g, $|G|$ has the maximum value when (1.3.3) has the minimum possible positive value. By simple arithmethical calculations one can show that the minimum positive value of (1.3.3) is attained when $\gamma = 0$, $m_1 = 2$, $m_2 = 3$, $m_3 = 7$ and the corresponding maximum value of G is 84(g - 1), and this is the famous result of Hurwitz.

Further the famous minimum genus problem is that for a finite group G we obtain a minimum possible values of g($\geq$ 2) called the minimum genus of a compact Riemann surface admitting G as its automorphism group.

1.5 Some important and interesting results

We now mention some important results in this field which have been proved in the last century by different mathematicians. During the last forty years or so the following broad problems on Riemann surface automorphism groups have been extensively studied and many beautiful results obtained. As mentioned earlier a Hurwitz group is representable as a group of 84(g - 1) automorphisms of a compact Riemann surface of genus g and this is the largest group that a compact Riemann surface of genus g can admit as an automorphism group. The second and third largest groups acting as automorphism groups of a compact Riemann surface of genus g can have order 48(g-1) and 40(g-1) respectively. The second and third largest groups which act as automorphism groups of compact Riemann surface of genus $g \geq 2$ are referred to as M_2-group and M_3-group respectively. The problem of finding all large groups of automorphisms including the Hurwitz group attracted the attention of the mathematicians in the last century.

Macbeath [96] derived a number of beautiful results about Riemann surface automorphism groups showing in particular that the bounds of Hurwitz [77] and Wiman [133] are attained infinitely many times. He also proved that an infinite number of the projective special linear groups are representable as Hurwitz groups, that is groups of $84(g-1)$ automorphisms of a compact Riemann surface of genus $g \geq 2$.

M.D.E. Conder [37, 38] proved that for infinitely many positive values of n, A_n, eleven of the twentysix sporadic groups are all Hurwitz groups. Marston Conder and Bujalance [18] studied the situation when a cyclic group G is the image of a finitely maximal Fuchsian group Γ with torsion-free Kernel K and whether or not G is the full group of automorphisms of some compact Riemann surface of genus $g \geq 2$.

Suzuki [131] proved that all simple Ree groups $2G_{2(q)}$ for $q = 3^{2m+1}$ whose order is $q^3(q^3 + 1)(q - 1)$ and $2F_{4(q)}$ for $q = 2^{2m+1}$ whose order $= q^{12}(q^6 + 1)(q^4 - 1)(q^3 + 1)(q - 1)$ are Hurwitz groups.

Chetiya [23] [24] proved that infinitely many solvable groups of derived length 3 or 4 occur as groups of automorphisms of the second largest order. The bound $48(g-1)$ for this family of groups is referred to as ***Chetiya bound*** by Gromadzki [63].

Chetiya and Kalita [32] also discovered many families of large solvable automorphism groups which have derived length at most 4. In [33] it is found that infinitely many of PGL(2, q), PSL(2, q), some S_n and A_n occur as M_2-groups and M_3-groups [37] [38].

Another interesting problem is to determine the bounds for different classes of finite groups. This problem is referred to as the upper bound problem.

Chetiya and Patra [34] considered the upper bound problem for the class of metabelian groups and the first three bounds for this class were obtained as $24(g - 1)$, $20(g - 1)$ and $16(g - 1)$ with the third bound being attainable for infinitely many values of g. The first and the second bounds are attained only for $g = 3$ and 5 respectively. The corresponding Fuchsian group being (3, 3, 4) and (2, 5, 5).

Determination of minimum genus of a surface on which a given finite group G acts as a group of automorphisms is another very interesting and general problem to solve. The problem, usually referred to as the minimum genus problem in the theory of Riemann surface automorphism groups, in its generality is far from being solved. Harvey [70] determined the minimum genus of a compact Riemann surface which admits a cyclic group as its automorphism groups. The same problem for the class of non-cyclic abelian groups, K-metacyclic groups, Zs-metacyclic groups, PSL(2, p) groups was solved respectively by C. Maclachlan [102], Chetiya and Patra [35] and Chetiya, Dutta and Patra [26], Glover and Sjerve [57].

Harvey [70] proved that if C_n is a cyclic group of order n and the prime decomposition of n is given by $n = p_1^{r_1} p_2^{r_2} \dots p_s^{r_s}; p_1 < p_2 < \dots < p_s$, then the minimum genus g of the surface admitting C_n as a group of automorphism is given by

$$\text{(a)}\quad g = \max\left\{2, \left(\frac{p_1 - 1}{2}\right)\frac{n}{2}\right\} \text{ if } r_1 > 1 \text{ or if n is a prime;}$$

$$\text{(b)}\quad g = \max\left\{2, \left(\frac{p_1 - 1}{2}\right)\left(\frac{n}{p_1} - 1\right)\right\} \text{ if } r_1 = 1.$$

Wiman's result follows immediately from that of Harvey.

Zomorrodian [136] considered the upper bound problem for a class of nilpotent groups. He proved the following result :

If G is a nilpotent group of automorphisms of a compact Riemann surface of genus $g \geq 2$, then O(G), the order of G $\leq 16(g - 1)$. If $O(G) = 16(g - 1)$, then $g - 1$ is a power of 2. Conversely, if $g - 1$ is a power of 2 then there exists at least one compact Riemann surface S of genus $g \geq 2$ with an automorphism group of order $16(g - 1)$ which must be nilpotent since its order is a power of 2. This bound corresponds to the Fuchsian group (2, 4, 8) and given any non-negative integer n, there exists a compact Riemann surface of genus $g = 2^n + 1$ that admits an automorphism group of order $16(g - 1) = 2^{n+4}$. He also studied the family of p-groups [133] where p is a prime ≥ 3. The result is : If a finite p-group G occurs as an automorphism group of a compact Riemann surface of genus $g \geq 2$, then

$$O(G) \leq \begin{cases} 9(g-1) \text{ if } p=3, \\ \dfrac{2p(g-1)}{p-3} \text{ if } p \geq 5 \cdot \end{cases}$$

Moreover, if $g - 1 = 3^n$, $n \geq 2$ or if $g-1=\frac{(p-3)p^n}{2}$ and $p \geq 5$, $n \geq 0$, then there is a compact Riemann surface of genus g that admits a group of order 3^n or p^{n+1} as a group of automorphisms.

G. Gromadzki and Maclachlan [64] solved the upper bound problem for super solvable groups showing that a super solvable group of automorphisms of a compact Riemann surface $g \geq 3$ had no more than $18(g - 1)$ elements. For $g = 2$, a super solvable group of order 24 acts as an automorphism group of a compact Riemann surface of genus g. Furthermore, it was proved that a necessary and sufficient condition for the existence of a compact Riemann surface of genus $g \geq 3$ that admits a supersolvable group of automorphisms of order $18(g - 1)$ is that 3^2 divides $g - 1$ and the only prime divisors of $g - 1$ are congruent to 1 mod 3.

Chetiya and Patra [35] proved that the minimum genus g of a compact Riemann surface admitting a K-metacyclic group of order $p(p - 1)$ as a group of automorphisms is (i) 2, 4 or 8 according as $p = 3$, 5 or 7 and (ii) $1+\frac{p(p-7)}{4}$ or $1+\frac{p(p-5)}{4}$ according as $p \equiv - 1 (\text{mod } 4)$ or $p \equiv 1 \ (\text{mod } 4)$ when $p \neq 3, 5, 7$.

Gromadzki Grzegarz [66] showed that a Riemann surface of even genus g admits at most four conjugacy classes of symmetries.P. Ahmed [9] in her doctoral thesis solved the minimum genus problem in the theory of Riemann surface automorphism groups in relation to the generalized quaternion group, which constitute an important subclass of the metacyclic groups. The upper bound of a generalized quaternion group Q_{4m} acting as an automorphism group of a compact Riemann surface of genus g is 12(g - 1) and the bound is attained infinitely many times if m is odd and a multiple of 3. The next two possible bounds in order of magnitude are 8(g - 1) and $\frac{20}{3}(g-1)$ and these bounds are attained infinitely many times for even values of m and values of m having 5 as the smallest prime divisor respectively. Another subclass of non-abelian metacyclic group of order pq, p and q being primes, q | p - 1, was considered by Chetiya, Dutta and Patra [26]. The following result was obtained by them :

If G is a non-abelian group of order pq, where p and q are primes and q | p - 1 then the minimum value of the genus g of a compact Riemann surface on which G acts as an automorphism group is given by :

(i) $g = p - 1$ for $q = 2$.

(ii) $g=\frac{1}{2}(p-1)$ for $q = 3$

(iii) $g=1+\frac{pq}{2}\left(1-\frac{3}{q}\right)$ for $q > 3$.

To find the complete set of genera of the surface on which a given finite group G acts as an automorphism group is referred to as the ***genera problem***. Chetiya, Dutta and Patra [28] solved the genera problem for a class of strictly metabelian groups. They considered the class of Dihedral groups of order 2p (which they denoted by D_{2p}), p being an odd prime. They established the following result :

Let p be an odd prime and D_{2p} the Dihedral group of order 2p. Then there exists a compact Riemann surface of genus g on which D_{2p} acts effectively as a group of automorphisms if and only if the integer g can be expressed as $g = \lambda + \mu p$ where λ, μ are integers with $\lambda \leq 1$ and $\mu \geq |\lambda|$ and λ, μ are not simultaneously zero.

Kulkarni [86], Kulkarni and Maclachlan [89], MeCullough and Miller [115] took the initiative by solving the genera problem for some special types of cyclic and non-cyclic abelian groups.

In 1987, R. Kulkarni [86] proved the following beautiful result :

For every finite group G there is a divisor N(G) of the order of G such that,

(i) If G acts as an automorphism group of a compact Riemann surface of genus g then N(G) is a divisor of g - 1;

(ii) If N(G) is a divisor of g - 1, and g is not one of a finite set of exceptional numbers then there is a compact Riemann surface of genus g on which G acts as an automorphism group.

In 1995 R. S. Kulkarni [88] using topological method investigated compact Riemann surfaces admitting 'large' automorphism groups, where a group G is defined to be large if its order is strictly greater than 4(g - 1). Here the classical result of Harvey and Wiman is elucidated by classifying Riemann surfaces admitting large cyclic automorphism groups.

In 1990, Macbeath [100] described an algorithm which, for a given integer $N \geq 2$, enables one to list all Riemann surface automorphism groups with the genus g of the surface satisfying $2 \leq g \leq N$.

Besides these recently Chutiya [48] in his doctoral thesis considered an important subclass of metabelian group i.e. a Dihedral group D_{2n} of order 2n for every positive integer n.

We now mention some more significant and interesting recent results in this field. Chetiya and Dutta [30] obtained the genera of compact Riemann surfaces admitting successive triangular extensions of a class of triangular groups as their automorphism groups. Hayakawa, Keizo and Kuribayashi, Akikazu [76] moulded the work of W. Burnside [20] in new grab by studying orientation-preserving automorphism groups of finite order of a compact Riemann surface of genus one. Yang, Qingjie [135] proved that if S is a compact Riemann surface of genus $g > 1$, G a group of analytic automorphisms of S which can be represented as a subgroup of R(S, G) of $GL_g(C)$, then a dihedral subgroup of order 2p in $GL_g(C)$, p an odd prime is realized by some S and some G if and only if each non-identity element has integer trace less than or equal to 1. Bogopol'skii, O. V. [21] classified the actions of finite groups on orientable surfaces of genus 4. Zimmerman, Jay [140] and Zimmerman, Jay and May, Coy L. [141] defined the minimum genus as symmetric genus. They [140] [141] found the symmetric genus of all groups of order 32. They [140] [141] also classified the groups of symmetric genus 3 and showed that the groups $Z_2 \times Z_2 \times S_4$, PSL(2, 7) and PGL(2, 7) are the only three groups having symmetric genus 3.

The problems and their solutions mentioned above apply basically to birational transformations of complex algebraic curves.

1.6 A brief discussion of the thesis

The purpose of this thesis is to study some families of non-commutative finite groups with small sets of generators whose orders are also comparatively small. In our study of these groups we exploit the well known fact [20] [59] that every finite group can be represented as a group of automorphisms (biholomorphic self transformations) of a compact Riemann surface of genus $g \geq 2$. But in the study of Riemann surface automorphisms the theory of Fuchsian groups comes into play. The theory of Fuchsian groups has been successively used [24] [25] [34] in determining the automorphism groups of compact Riemann surface of genus $g \geq 2$.

We now give the outline of the thesis. In chapter 2 we consider some admissible triples like (ℓ, m, n) imposing in each case some conditions among the integers ℓ, m, n. In each case we prove the existence of an infinite number of solvable smooth quotients with short derived series. We also present some theorems and lemmas for references and use it in the later chapters.

The main results proved in this chapter are contained in the following theorems :

Theorem 2.1.1 [50] : Let $\Gamma = (\ell, m, n)$ be a Fuchsian group where ℓ, m, n are positive integers ≥ 2 such that $(\ell, m) = d > 1$ and $(\ell, n) = (m, n) = 1$, ℓ, m, n donot simultaneously assume the values $\ell = m = 2$ or $\ell = m = 3$, $n = 2$. Then for each positive integer $k \geq 1$, $\Gamma = (\ell, m, n)$ has a solvable smooth quotient G_s of derived length 3 and of order $(kt)^{2\gamma''} dn^{d-1} a^{n^{(d-1)}-1} b^{n^{(d-1)}-1}$.

Theorem 2.1.2 : Let $\Gamma = (\ell, m, n)$ be a Fuchsian group where ℓ, m, n are positive integers ≥ 2 such that $(\ell, m) = d_1 > 1$, $(\ell, n) = d_2 > 1$, $(m, n) = 1$ and $(d_1, d_2) = 1$ (as $(m, n) = 1$). Then for each positive integer $k > 1$, Γ admits smooth quotients of order :

$$d_1 d_2 k^{2\gamma'} t^{2\gamma_k} a^{A+2\gamma_k-1} b^{A/h_1+2\gamma_k-1} c^{A/h_2+2\gamma_k-1} h_1^{d_2-A/h_1-2\gamma_k} h_2^{d_1-A/h_2-2\gamma_k}, k > 1$$

and of genus :

$$\frac{1}{2h_1^{A/h_1+2\gamma_k-1} h_2^{A/h_2+2\gamma_k-1}} \left(a^{A+2\gamma_k-1} b^{A/h_1+2\gamma_k-1} c^{A/h_2+2\gamma_k-1} t^{2\gamma_k} \right) \Bigg[2\gamma_k - 2$$

$$+ A\left(1 + \frac{1}{h_1} + \frac{1}{h_2} - \frac{1}{a} - \frac{1}{b} - \frac{1}{c} \right) \Bigg] + 1,$$

where $A = h_1^{d_2-1} h_2^{d_1-1} k^{2\gamma'}$.

Theorem 2.1.3 : Let $\Gamma = (\ell, m, n)$ be a Fuchsian group where ℓ, m, m are positive integers ≥ 2 such that $(\ell, m, n) = r \geq 1$ and $(\ell, m) = rd_1$ $(\ell, n) = rd_2$, $(m, n) = rd_3$ for some non-negative integers d_1, d_2 and d_3, where d_1, d_2, d_3 are pair-wise prime to each other. Then Γ admits a metabelian smooth quotient of order,

$r^2 d_1^2 d_2 d_3 k^{2\gamma'} a^{rd_1d_3+2\gamma'-1} b^{rd_1d_2+2\gamma'-1} c^{rd_1^2+2\gamma'-1}$ and of genus,

$$\frac{1}{2}\left(a^{rd_1d_3+2\gamma'-1} b^{rd_1d_2+2\gamma'-1} a^{rd_1^2+2\gamma'-1} k^{2\gamma'} \right) \Bigg[2\gamma' - 2 +$$

$$+ rd_1 \left(d_1 + d_2 + d_3 - \frac{d_3}{a} - \frac{d_2}{b} - \frac{d_1}{c} \right) \Bigg] + 1.$$

[In case $r = 1$, then at least one of d_1, d_2, d_3 is not equal to 1 because in that case (ℓ, m, n) will be a perfect group.]

In chapter 3, we introduce a class of Zs-metacyclic groups and derive a set of necessary and sufficient conditions for the existence of a smooth homomorphism from a Fuchsian group onto a Zs-metacyclic group of order $4p^2$, p an odd prime. Using these necessary and sufficient conditions we determine the minimum genus of a surface on which a Zs-metacyclic group of order $4p^2$ acts as a group of automorphisms of compact Riemann surface. The necessary and sufficient conditions are given in the following theorem :

Theorem 3.1.1 : Let $\Gamma = (\gamma; m_1, ..., m_s)$, $s \geq 0$ be a Fuchsian group of genus γ, there exists a smooth epimorphism from Γ to a Zs-metacyclic group M where,

$$M = \left\langle a, b : a^{p^2} = b^4 = 1, b^{-1}ab = a^{-1} \right\rangle$$

and is of order $4p^2$, p an odd prime ≥ 3 if and only if the periods of Γ satisfy the following conditions :

(1) If Γ is a surface group then $\gamma \geq 2$.

(2) If Γ is not a surface group then the periods of Γ must take the values from the set $\{2, 4, p, p^2, 2p, 2p^2\}$.

(3) If t_2, t_4, t_p, t_{p^2}, t_{2p} and t_{2p^2} denote respectively the number of periods of order 2, 4, p, p^2, 2p and $2p^2$ then

(A) t_4 must be even.

(B) If s = 1 then (i) $\Gamma = (\gamma; p)$; $\gamma \geq 2$

or (ii) $\Gamma = (\gamma; p^2)$; $\gamma \geq 1$.

(C) If s = 2 then Γ has the following signatures :

(i) $(\gamma; 2, 2)$; $\gamma \geq 2$.

(ii) $(\gamma; 2, 2p)$; $\gamma \geq 2$.

(iii) $(\gamma; 2, 2p^2)$; $\gamma \geq 1$.

(iv) $(\gamma; 4, 4)$; $\gamma \geq 1$.

(v) $(\gamma; p, p)$; $\gamma \geq 2$.

(vi) $(\gamma;\ p,\ p^2)$; $\gamma \geq 1$.

(vii) $(\gamma;\ p^2,\ p^2)$; $\gamma \geq 1$.

(viii) $(\gamma;\ 2p,\ 2p)$; $\gamma \geq 2$.

(ix) $(\gamma;\ 2p,\ 2p^2)$; $\gamma \geq 1$.

(x) $(\gamma;\ 2p^2,\ 2p^2)$; $\gamma \geq 1$.

(D) If $s \geq 3$ and $t_4 = 0$ then $\gamma \geq 1$.

(a) If t_4, t_{p^2}, t_{2p^2} are simultaneously zero then $\gamma \geq 2$.

(b) If 'i' is the only period of Γ, then t_i can not take odd values for the following values of i :

$i = 2$, $2p$ and $2p^2$.

(c) (i) If i, j, k$(i \neq j \neq k)$ are the three periods of Γ then t_i, t_j, t_k cannot take odd values simultaneously for the following values of i, j, k :

$i = 2$, $j = 2p$, $k = 2p^2$.

(ii) If there are two periods of Γ instead of three taking values from the set $\{2, 2p, 2p^2\}$ then the number of them will be both even or both odd.

(E) If $s \geq 3$, $t_4(\neq 0) = 2$ and

(i) t_{p^2}, t_{2p^2} are simultaneously zero then $\gamma \geq 1$ otherwise $\gamma \geq 0$.

(ii) The conditions mentioned in 3(D) (b), (c) will also prevail in this case $[t_4 = t_4' + t_4'' \neq 0]$. In this case γ may be greater than or equal to zero. But when t_{p^2} and t_{2p^2} both are simultaneously zero then $\gamma \geq 1$.

Applying the conditions given in the theorem above we obtain our main results of this chapter which is given in the form of a theorem :

Theorem 3.2.1 : Let M be a Zs-metacyclic group of order $4p^2$ where p is an odd prime, with presentation:

$$\left\langle a, b : a^{p^2} = b^4 = 1, b^{-1}ab = a^{-1} \right\rangle$$

which act as an automorphism group of some compact Riemann surface of genus $g \geq 2$. Then the minimum value of g and the corresponding signature of the Fuchsian group Γ of which M is a smooth quotient is :

$$g = p^2 - 2p + 1; \ (4, 4, p).$$

In the last chapter i.e. in chapter 4, we study some Fuchsian quadruple groups like (m_1, m_2, m_3, m_4) where m_1, m_2, m_3 and m_4 are positive integers > 1 imposing certain conditions among them. Here we present a method of construction of finite smooth quotients with short derived series.

We also obtain the genera of compact Riemann surfaces admitting successive quadruple extensions of a class of quadruple groups as their automorphism groups.

The main results of this chapter are given in the following theorems :

Theorem 4.1.1 : Let $\Gamma = (\ell, m, n, \mu)$ be a Fuchsian group where ℓ, m, n, μ are positive integers greater than or equal to 2 such that, $(\ell, m) = d_1 > 1$, $(\ell, n) = (\ell, \mu) = (m, n) = (m, \mu) = (n, \mu) = 1$. Then Γ admits a solvable smooth quotient of derived length 3 and of order $k^{2\gamma''} d_1 (n\mu)^{d_1-1} (ab)^{(n\mu)^{d_1-1}-1+2\gamma''}$ and genus

$$\frac{1}{2} k^{2\gamma''} (ab)^{(n\mu)^{d_1-1}-2+2\gamma''} \left[2ab(\gamma''-1) + (n\mu)^{d_1-1}(2ab-b-a) \right] + 1, k \geq 1$$

where $$\gamma'' = \frac{n^{d_1-1}\mu^{d_1-1}}{2}\left[2d_1 - 2 - \frac{d_1}{n} - \frac{d_1}{\mu} \right] + 1.$$

Theorem 4.1.2 : Let $\Gamma = (\ell, m, n, \mu)$ be a Fuchsian group where ℓ, m, n, μ are positive integers greater than or equal to 2 such that, $(\ell, m) = d_1 > 1$, $(\ell, n) = d_2 > 1$ and $(\ell, \mu) = (m, n) = (m, \mu) = (n, \mu) = 1$ where d_1, d_2 are prime to each other. Then Γ admits a finite smooth quotient of order $d_1 d_2 AB$ where,

$$A = h_1^{d_2(d_1+1)-2} h_2^{d_1(d_2+1)-2} k^{2\gamma'}; k, h_1, h_2 \geq 1$$

$$B = a^{A-1}\left(\frac{b}{h_1}\right)^{A/h_1 - 1}\left(\frac{c}{h_2}\right)^{A/h_2 - 1}\left(\frac{\mu}{h_3}\right)^{A/h_3 - 1}(t\ell')^{2\gamma_k},$$

where $\ell' = \dfrac{abc\mu}{h_1 h_2 h_3}$;

and of genus,

$$\frac{B}{2}\left[2\gamma_k - 2 + A\left(1 + \frac{1}{h_1} + \frac{1}{h_2} + \frac{1}{h_3} - \frac{1}{a} - \frac{1}{b} - \frac{1}{c} - \frac{1}{\mu}\right)\right] + 1.$$

Theorem 4.1.3 : Let (ℓ, m, n, μ) be a Fuchsian group where ℓ, m, n, μ are positive integers greater than or equal to 2 such that, $(\ell, m) = d_1 > 1$, $(\ell, n) = d_2 > 1$, $(\ell, \mu) = d_3 > 1$ and $(m, n) = (n, \mu) = (m, \mu) = 1$, where d_1, d_2, d_3 are prime to each other. Then Γ admits a finite smooth quotient of Γ of genus :

$$\frac{B}{2}\left[2\gamma_k - 2 + A\left(1 + \frac{1}{h_1} + \frac{1}{h_2} + \frac{1}{h_3} - \frac{1}{a} - \frac{1}{b} - \frac{1}{c} - \frac{1}{r}\right)\right] + 1$$

and the order is $d_1 d_2 d_3 AB$,

where $A = h_1^{d_2 d_3 - 1} h_2^{d_1 d_3 - 1} h_3^{d_1 d_2 - 1} k^{2\gamma'}$,

$$B = a^{A-1}\left(\frac{b}{h_1}\right)^{A/h_1 - 1}\left(\frac{c}{h_2}\right)^{A/h_2 - 1}\left(\frac{r}{h_3}\right)^{A/h_3 - 1}(t\ell')^{2\gamma_k},$$

$$\ell' = \frac{abcr}{h_1 h_2 h_3}$$

and $$\Gamma_k = \frac{A}{2}\left[2\gamma' - 2 + d_2 d_3\left(1 - \frac{1}{b}\right) + d_1 d_3\left(1 - \frac{1}{c}\right) + d_1 d_2\left(1 - \frac{1}{r}\right)\right.$$
$$\left. - \frac{1}{h_1} - \frac{1}{h_2} - \frac{1}{h_3} + \frac{1}{b} + \frac{1}{c} + \frac{1}{r}\right] + 1.$$

Theorem 4.1.4 : Let $\Gamma = (\ell, m, n, \mu)$ be a Fuchsian group where ℓ, m, n, μ are positive integers ≥ 2 such that $(\ell, m) = rd_1$, $(\ell, n) = rd_2$, $(m, n) = rd_3$, $(\ell, m, n) = r \geq 1$ and $(\ell, \mu) = (m, \mu) = (n, \mu) = 1$, where d_1, d_2, d_3 are pairwise prime to each other. Then Γ admits a metabelian smooth quotient of order $Ar^2d_1{}^2d_2d_3$ and of genus :

$$\frac{A}{2}\left[2r^2d_1^2d_2d_3 - \frac{rd_1d_3}{a} - \frac{rd_1d_2}{b} - \frac{rd_1^2}{c} - \frac{r^2d_1^2d_2d_3}{\mu}\right] + 1,$$

where $A = a^{rd_1d_3+2\gamma'-1}b^{rd_1d_2+2\gamma'-1}c^{rd_1^2+2\gamma'-1}\mu^{r^2d_1^2d_2d_3+2\gamma'-1}k^{2\gamma'}, k \geq 1$

and $\gamma' = \frac{1}{2}r\left[rd_1^2d_2d_3 - d_1d_3 - d_1d_2 - d_1^2\right] + 1$.

Theorem 4.1.5 : Let $\Gamma = (\ell, m, n, \mu)$ be a Fuchsian group where ℓ, m, n, μ are positive integers greater than or equal to 2 such that, $(\ell, m) = d_1 > 1$, $(\ell, n) = d_2 > 1$, $(\ell, \mu) = d_3 > 1$, $(m, n) = d_4 > 1$ and $(m, \mu) = (n, \mu) = 1$, where d_1, d_2, d_3, d_4 are pairwise prime to each other. Then Γ admits a metabelian smooth quotient of order $Ad_1d_2{}^2d_3d_4$ and of genus,

$$\frac{A}{2}\left[2d_1d_2^2d_3d_4 - \frac{d_2d_4}{a} - \frac{d_2^2d_3}{b} - \frac{d_1d_2d_3}{c} - \frac{d_1d_2^2d_4}{d}\right] + 1,$$

where $A = a^{d_2d_4+2\gamma'-1}b^{d_2^2d_3+2\gamma'-1}c^{d_1d_2d_3+2\gamma'-1}d^{d_1d_2^2d_4+2\gamma'-1}k^{2\gamma'}$, $k \geq 1$

and $\gamma' = \frac{1}{2}[2d_1d_2^2d_3d_4 - d_2d_4 - d_2^2d_3 - d_1d_2d_3 - d_1d_2^2d_4] + 1$.

In the last section of this chapter we find the genus of the p(i)-th extension G_i of G by taking a general quadruple group (ℓ, m, n, ξ) where ℓ, m, n, ξ are positive integers greater than or equal to 2. The result is given in the form of a theorem :

Theorem 4.3.1 : Let G be an admissible quadruple group of type (ℓ, m, n, ξ)

generated by u, v, w such that u, v, w and uvw have ℓ, m, n and ξ as their respective orders. Let $P = \{p_1, \ldots, p_k\}$ be a set of k-distinct primes such that $p_1 < \ldots < p_k$ and p_i not divide m_{i-1}, n_{i-1} and ξ_{i-1} where m_i, n_i and ξ_i are integers defined as follows :

(i) $m_0 = m$, $m_1 = [m_0, p_1]$, $m_2 = [m_1, p_2]$, ...

(ii) $n_0 = n$, $n_1 = [n_0, p_1]$, $n_2 = [n_1, p_2]$, ...

(iii) $\xi_0 = \xi$, $\xi_1 = [\xi_0, p_1]$, $\xi_2 = [\xi_1, p_2]$, ...

where [a, b] denotes the l.c.m. of a and b.

Then p(i)-th extension G_i of G is an automorphism group of a compact Riemann surface of genus g_i, $1 \leq i \leq k$ where

$$g_i = 1 + \frac{1}{2}(p_1 \ldots p_i)\,|G|\left\{2 - \frac{1}{\ell} - \frac{1}{m_i} - \frac{1}{n_i} - \frac{1}{\xi_i}\right\}.$$

We conclude our thesis with a fairly exhaustive bibliography on the topic of our study.

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Chapter 2 : Finite solvable smooth quotient of some two generator groups

2.1 Existence of solvable smooth quotients

2.2 Conclusion

CHAPTER - 2

FINITE SOLVABLE SMOOTH QUOTIENTS OF SOME TWO GENERATOR GROUPS

A finite group G can be represented as a group of automorphisms of a compact Riemann surface. In this chapter we study some finite groups that can be generated by two elements and which acts as Riemann surface automorphism groups. We begin this chapter by first defining what we mean by two generator group.

Cyclic groups are the groups having the minimum number of generators. These are the one generator groups. The number of generators of a non-cyclic group is greater than one. Hence a non-cyclic group has atleast two generators. The finite groups which can be generated by two generators only are called ***two-generator*** groups. All commutative two generator groups are known. It is a problem of considerable importance and interest in group theory to determine all non-commutative two generator groups, mainly because of its relation to finding similar presentation of groups. There are many known families of non-commutative two generator finite groups. A good number of these is contained in Coxeter and Moser [47] and Johnson [78].

A finite group G will be called a ***triangle group*** of type (ℓ, m, n) where ℓ, m, n are positive integers if G can be generated by two elements x, y satisfying the relations :

$$x^{\ell} = y^m = z^n = xyz = 1$$

or $$x^{\ell} = y^m = (xy)^n = 1.$$

A triangle group (ℓ, m, n) is called ***admissible*** if $\frac{1}{\ell}+\frac{1}{m}+\frac{1}{n}<1$.

By section 1.3 of chapter 1, a Fuchsian triangle group is of the form (m_1, m_2, m_3) having three periods and genus $\gamma = 0$. A Fuchsian triangle group can be generated by two elements x_1, x_2 satisfying :

$$x_1^{m_1} = x_2^{m_2} = (x_1x_2)^{m_3} = 1$$

and $$\frac{1}{m_1}+\frac{1}{m_2}+\frac{1}{m_3}<1.$$

We know that a finite group G is a group of automorphisms of a compact Riemann surface of genus $g \geq 2$ if and only if there exists an epimorphism ϕ from Γ onto G which preserves the periods of Γ and we call G a smooth quotient of Γ. So an admissible triangle group (ℓ, m, n) will be a group of automorphisms of a compact Riemann surface S of genus $g \geq 2$ if and only if it is a smooth quotient of a Fuchsian triangle group (ℓ, m, n) where

$$\frac{1}{\ell}+\frac{1}{m}+\frac{1}{n}<1.$$

We must look at proper quotients of Fuchsian triangle groups to get solvable groups. Also smooth quotients of Fuchsian groups are particularly interesting for their connection with Riemann surface automorphism groups. Therefore the study of solvable smooth quotients of Fuchsian triangle groups is of special interest.

Macbeath showed [96] that maximal automorphism group of compact Riemann surfaces occur as quotients of Fuchsian triangle group having periods 2, 3 and 7 and these groups are perfect groups. Similarly Chetiya [23] in his Ph. D. thesis showed that the maximal solvable automorphism groups are also quotients of a Fuchsian triangle group having periods 2, 3 and 8. A quotient of a Fuchsian triangle group is a two generator group. This shows the importance of finding quotients of different classes of Fuchsian triangle group from a strictly group theoretic point of view. This was the theme of the papers by Chetiya [24] [25], Chetiya and Kalita [32], Chetiya, Dutta and Patra [27], where the existence of several infinite classes of two generator finite solvable automorphism groups of compact Riemann surfaces was proved.

In this chapter we want to generalize these results for any Fuchsian triangle group $\Gamma = (\ell, m, n)$. As we are interested in finding solvable quotients of Fuchsian triangle group, we consider those Fuchsian triangle groups whose quotients are ***non-perfect***. It may be noted that a ***non-perfect*** quotients of Fuchsian triangle group contains atleast ***two periods*** which are not ***relatively prime***.

In the course of the development of the contents of this chapter we need to find the signature of some normal subgroups of finite index of Fuchsian groups. A general method of finding the signature of a subgroup of finite index of a Fuchsian group is described by Singerman in [124]. Maclachlan [102] also devised a method of obtaining the signature of a normal subgroup of finite index of general Fuchsian group by using the findings of Macbeath [96] and Knopp and Newman [85]. This result of Maclachlan was used by Bujalance, Gamboa and Gromadzki [15] for compact Fuchsian groups. We shall use a version of Maclachlan's result and include this result (without proof) in the form of a Lemma (2.1.1).

We now proceed to find an infinite family of finite solvable smooth quotients of the Fuchsian triangle group (ℓ, m, n).

In our discussion let us agree to denote the h.c.f. of the numbers n_1, n_2, n_3 by (n_1, n_2, n_3). Also if the numbers n_1, n_2 and n_3 are pairwise (or relatively) prime then $(n_1, n_2, n_3) = 1$, but not conversely. We can now turn to the actual analysis of the admissible triples. We divide the admissible triples broadly into the following classes :

Class I : (ℓ, m, n) where $(\ell, m) = d > 1$ and $(\ell, n) = (m, n) = 1$.

Case II : (ℓ, m, m) where $(\ell, m) = d_1 > 1$, $(\ell, n) = d_2 > 1$, $(m, n) = 1$.

Case III : (ℓ, m, n) where $(\ell, m, n) = r \geq 1$, $(\ell, m) = rd_1$, $(\ell, n) = rd_2$, $(m, n) = rd_3$.

In case $r = 1$, at least one of d_1, d_2, d_3 is not equal to 1 because then (ℓ, m, n) will be a perfect group.

The periods ℓ, m, n are taken as unordered. All the above three classes will be stated latter on in the form of theorems.

2.1 Existence of solvable smooth quotients

We start this section with a definition followed by four lemmas which will be used in proving our main theorems.

Definition 2.1.1 : Let K be any group. For each integer n, K_n^* is defined to be the subgroup of K generated by the n^{th} powers of the elements of K. In what follows we shall use the notation $K^{(i)}$ to denote the i^{th} derived group of a group K.

Let Γ be a Fuchsian group having signature $\Gamma = (\gamma; m_1, ..., m_k)$, $k \geq 2$, then the periods of Γ are said to satisfy the ***l.c.m. condition*** if the $[m_1,...,\hat{m}_i, m_{i+1},...,m_k] = [m_1,...,m_i,...,m_k]$ for all i, $1 \leq i \leq k$, where $\hat{m}_i$ denotes omission of m_i. It may be noted that [a, b] denotes the l.c.m. of a and b.

For satisfaction of the l.c.m. condition, the group must have at least two periods. If the periods of a Fuchsian group satisfy the l.c.m. condition then the next derived group must be a surface group [99].

We now give Maclachlan's version without proof for finding the signature of normal Fuchsian subgroup of finite index in Lemma 2.1.1. This Lemma will be followed by 3 more Lemmas : Lemma 2.1.2, Lemma 2.1.3, Lemma 2.1.4. The proof of these Lemmas are given in this section.

Lemma 2.1.1 [15] [107] : Let $\Gamma = (\gamma; m_1, .., m_k)$ be a Fuchsian group with generators $x_1, .., x_k$ of finite order and $\alpha_1, \beta_1, ..., \alpha_\gamma, \beta_\gamma$ of infinite order. Let N be a normal subgroup of Γ of finite index. Let p_i be the order of the image of x_i in the quotient Γ/N and let $I = \{1 \leq i \leq k; m_i \neq p_i\}$. Also let $n_i = m_i/p_i$ and $s_i = [\Gamma : N]/p_i$ for every $i \in I$ then $N = (\gamma'; n_i, ..., n_i)$, $i \in I$ where each n_i occurs s_i times and γ' is obtained from :

$$[\Gamma : N] = \frac{\delta(N)}{\delta(\Gamma)}. \qquad ......(2.1.1)[73]$$

The proof is omitted.

Lemma 2.1.2 [24] : Let K be a Fuchsian surface group of genus g and K', the derived group of K. If for each positive integer n, K_n^* denotes the subgroup of K generated by the n^{th} powers of all the generators of K, then the product $K_n = K_n^*K'$ is a normal surface subgroup of K such that $[K : K_n] = n^{2g}$.

Proof : Let K be generated by the elements $\alpha_1, \beta_1, ..., \alpha_g, \beta_g$. Then it is clear that K_n^*, the subgroup generated by the n^{th} powers of all the generators of K is a characteristic subgroup of K. Also since K', the derived group of K, is characteristic in K, the product $K_n = K_n^*K'$ is a characteristic subgroup of K. Further since $K' \subset K_n$, K/K_n is abelian. Let $a_1, b_1, ..., a_g, b_g$ be the images of $\alpha_1, \beta_1, .., \alpha_g, \beta_g$ under the natural homomorphism of K onto K/K_n. Now the quotient K/K' is a free abelian group of rank 2g, so K/K_n is generated by $a_1, b_1, ..., a_g, b_g$ with defining relations $a_1^n = b_1^n = ... = a_g^n = b_g^n = 1$.

Therefore $K/K_n \cong Z_n \oplus ... \oplus Z_n$ with 2g copies of the cyclic group Z_n and so $[K : K_n] = n^{2g}$.

Lemma 2.1.3 : Let Γ be a Fuchsian group of non-zero genus whose periods satisfy the l.c.m. condition and let 'ℓ' be the l.c.m. of the periods of Γ. If Γ' is the derived group of Γ and $\Gamma_{k\ell}$ is the subgroup of Γ generated by the '$k\ell$'-th power $(k \geq 1)$ of the generators of Γ, then $\Gamma_s = \Gamma_{k\ell}\Gamma'$ is a surface subgroup of Γ of finite index for any positive integer $k \geq 1$.

Proof : Let $\Gamma = (\gamma; m_1, .., m_s)$, $\gamma > 0$, $s \geq 2$, $m_i \geq 2$ be such that the periods of Γ satisfy the l.c.m. condition with l.c.m. of the periods of Γ being equal to 'ℓ'.

Let $x_1, ..., x_s$ be the finite order generators with orders $m_1, ..., m_s$ respectively and $\alpha_1, \beta_1, ..., \alpha_\gamma, \beta_\gamma$ be the infinite order generators of Γ satisfying the relations :

$$x_1^{m_1} = ... = x_s^{m_s} = \prod_{i=1}^{s} x_i \prod_{j=1}^{\gamma} [\alpha_j, \beta_j] = 1. \qquad ......(2.1.2)$$

As in Lemma 2.1.2, $\Gamma_s = \Gamma_K\Gamma'$ is a normal subgroup of Γ. Let $u_1, ..., u_s, a_1, b_1, ..., a_\gamma, b_\gamma$

be the images of $x_1, \ldots, x_s, \alpha_1, \beta_1, \ldots, \alpha_\gamma, \beta_\gamma$ respectively under the natural homomorphism of Γ onto Γ/Γ_s, then Γ/Γ_s is generated by $u_1, \ldots, u_s, a_1, b_1, \ldots, a_\gamma, b_\gamma$ satisfying the relations :

$$u_1^{m_1} = \ldots = u_s^{m_s} = \prod_{i=1}^{s} u_i \prod_{j=1}^{\gamma} [a_j, b_j] = 1. \qquad \ldots\ldots(2.1.3)$$

Since $\Gamma' \subseteq \Gamma_s$ so Γ/Γ_s is abelian and therefore the generators of Γ/Γ_s will commute with each other. Therefore (2.1.3) reduced to :

$$u_1^{m_1} = \ldots = u_s^{m_s} = u_1 \ldots u_s = 1. \qquad \ldots\ldots(2.1.4)$$

Further since $\Gamma_{k\ell} \subseteq \Gamma_s$ we have

$$u_1^{k\ell} = \ldots = u_s^{k\ell} = a_1^{k\ell} = b_1^{k\ell} = \ldots = a_\gamma^{k\ell} = b_\gamma^{k\ell} = 1. \qquad \ldots\ldots(2.1.5)$$

From (2.1.4) and (2.1.5) it follows that, the generators of Γ/Γ_s satisfy the following relations :

$$\left.\begin{array}{ll} & u_1^{m_1} = \ldots = u_{s-1}^{m_{s-1}} = (u_1 \ldots u_{s-1})^{m_s} = 1 \\ \text{and} & a_1^{k\ell} = b_1^{k\ell} = \ldots = a_\gamma^{k\ell} = b_\gamma^{k\ell} = 1. \end{array}\right\} \qquad \ldots\ldots(2.1.6)$$

From (2.1.6) we conclude that,

$$\Gamma/\Gamma_s \cong \underbrace{Z_{k\ell} \oplus \ldots \oplus Z_{k\ell}}_{2\gamma\text{-summands}} \oplus Z_{t_1} \oplus \ldots \oplus Z_{t_r},$$

where $r \leq s - 1$ and $t_i \in \{m_1, \ldots, m_s\}$.

Hence $[\Gamma : \Gamma_s] = (k\ell)^{2\gamma}\{t_1 \ldots t_r\}$ which is finite. We now apply Lemma 2.1.1 and we have, $p_1 = m_1, p_2 = m_2, \ldots, p_s = m_s$ and hence $n_1 = \ldots = n_s = 1$. Therefore Γ_s contains no finite order generators. Since any finite order element of a Fuchsian group is a conjugate of a power of a finite order generator, Γ_s contains no finite order element. Thus Γ_s is a surface group whose genus can be calculated from (2.1.1). This completes the proof.

Lemma 2.1.4 : Let $\Gamma = (\gamma; m, ..., m, n)$ be a Fuchsian group with $s - 1$ occurrences of m, $s \geq 2$, $m \geq 2$, $n \geq 2$, $\gamma \geq 1$ and $(m, n) = 1$. Then for any positive integer k, $k \geq 2$, Γ has a Fuchsian subgroup Γ_k whose periods satisfy the l.c.m. conditions and Γ_k is of index $d^{s-2}k^{2\gamma}$ where $d = (k, m)$ in Γ.

Proof : Let $x_1, ..., x_{s-1}$ be the generators of Γ of order m, x_s be the generator of order n and $\alpha_1, \beta_1, ..., \alpha_\gamma, \beta_\gamma$ be the generators of infinite order of Γ. Let Γ_k^* be the subgroup of Γ generated by the k^{th} powers of the generators of Γ, then $k \geq 2$ otherwise Γ_k^* coincides with Γ. It follows that Γ_k^* and Γ', the derived group of Γ, are characteristic subgroups of Γ and therefore the product $\Gamma_k = \Gamma_k^*\Gamma'$ is a characteristic subgroup of Γ and so a normal subgroup of Γ. Moreover $\Gamma' \subset \Gamma_k$ so Γ/Γ_k is abelian. Let $u_1, ..., u_{s-1}, u_s, a_1, b_1, ..., a_\gamma, b_\gamma$ be the images of $x_1, ..., x_{s-1}, x_s, \alpha_1, \beta_1, ..., \alpha_\gamma, \beta_\gamma$ respectively under the natural homomorphism of Γ onto Γ/Γ_k. Then we have,

$$u_1^m = ... = u^m_{s-1} = u_s^n = u_1 ... u_s = 1,$$

$$\text{and} \quad u_1^k = ... = u^k_{s-1} = u_s^k = a_1^k = b_1^k = ... = a_\gamma^k = b_\gamma^k = 1,$$

where the elements commute with each other.

As $(m, n) = 1$, the relations :

$$u_s^n = u_1 ... u_s = u_i^m = 1, \; i = 1, ..., s - 1 \text{ give } u_s = 1$$

$$\text{and} \quad u_{s-1} = (u_1 ... u_{s-2})^{-1}.$$

If $(k, m) = d$, $d \geq 1$, the relations :

$$u_i^m = u_i^k = 1, \; i = 1, ..., s - 2 \text{ gives}$$

$$u_i^d = 1 \text{ for } i = 1, ..., s - 2.$$

So $\Gamma/\Gamma_k \cong z_d \oplus ... \oplus z_d \oplus z_k \oplus ... \oplus z_k$ with $(s - 2)$ copies of cyclic group Z_d and 2γ copies of the cyclic group Z_k. Hence $[\Gamma : \Gamma_k] = d^{s-2}k^{2\gamma}$. Now by application of Lemma

(2.1.1) shows Γ_k has periods equal to n and m/d and as $k \geq 2$ and $\gamma \geq 1$, each of the periods are repeated at least twice and hence the periods of Γ_k satisfy the l.c.m. condition. This completes the proof.

We are now in a position to give the proofs of the theorems on the existence of solvable finite smooth quotients of Fuchsian triangle groups.

Theorem 2.1.1 [50] : Let $\Gamma = (\ell, m, n)$ be a Fuchsian group where ℓ, m, n are positive integers ≥ 2 such that $(\ell, m) = d > 1$ and $(\ell, n) = (m, n) = 1$. ℓ, m, n donot simultaneously assume the values $\ell = m = 2$ or $\ell = m = 3$, $n = 2$. Then for each positive integer $k \geq 1$, $\Gamma = (\ell, m, n)$ has a solvable smooth quotient G_s of derived length 3 and of order

$$(kt)^{2\gamma''} d n^{d-1} a^{n^{d-1}-1} b^{n^{d-1}-1}.$$

Proof : Let $\ell = ad$, $m = bd$, $d > 1$, $(a, b) = 1$, $a \geq 1$ and $b \geq 1$.

Let Γ be generated by elements x_1, x_2, x_3 satisfying :

$$x_1^{ad} = x_2^{bd} = x_3^{n} = x_1x_2x_3 = 1$$

or equivalently,

$$x_1^{ad} = x_2^{bd} = (x_1x_2)^n = 1.$$

Let u_1, u_2, u_3 be the images of x_1, x_2, x_3 respectively under the abelianizing homomorphism from Γ onto Γ/Γ'. Then Γ/Γ' is generated by u_1, u_2, u_3 satisfying :

$$u_1^{ad} = u_2^{bd} = u_3^{n} = u_1u_2u_3 = 1.$$

or $\quad u_1^{ad} = u_2^{bd} = (u_1u_2)^n = 1.$

The above relation gives $u_1^{ad} = u_1^{bd} = 1$.

But $(a, b) = 1$ implies $u_1^{d} = 1$.

So $\Gamma/\Gamma' \cong Z_d$.

By Lemma (2.1.1) $\Gamma' = \left(\gamma'; a, b, \underbrace{n, \ldots, n}_{d-\text{times}}\right)$.

From (2.1.1) we get $\gamma' = 0$.

So $$\Gamma' = \left(a, b, \underbrace{n, \ldots, n}_{d-\text{times}} \right). \qquad \text{......(2.1.1.1)}$$

Next let Γ' be generated by $y_1, \ldots, y_d, y_{d+1}, y_{d+2}$ satisfying :

$$y_1^n = y_2^n = \ldots = y_d^n = y^a_{d+1} = y^b_{d+2} = y_1 \cdots y_d \cdot y_{d+1} y_{d+2} = 1.$$

Then $\Gamma'/\Gamma'' \cong z_n \oplus \ldots \oplus z_n$ with (d - 1) summands.

Application of Lemma (2.1.1) gives,

$$\Gamma'' = \left(\gamma''; \underbrace{a, \ldots, a}_{n^{d-1}-\text{times}}, \underbrace{b, \ldots, b}_{n^{d-1}-\text{times}} \right). \qquad \text{......(2.1.1.2)}$$

From (2.1.1),

$$\gamma'' = \frac{1}{2} n^{d-2}[n(d-2)-d]+1. \qquad \text{......(2.1.1.3)}$$

$\gamma'' = 0$ only when d = 2 or d = 3, n = 2. For all d ≥ 4, n ≥ 3, we have $\gamma'' > 0$. So from (2.1.1.2) it is seen that the periods of Γ'' satisfy the l.c.m. condition. The l.c.m. of the periods being ab = t, say. Hence by Lemma (2.1.3) Γ'' has a normal surface subgroup $\Gamma_s = \Gamma_{kt}\Gamma''$ of finite index for every positive integer k ≥ 1.

Now $$[\Gamma'' : \Gamma_s] = (kt)^{2\gamma''} a^{n^{d-1}-1} b^{n^{d-1}-1}, k \geq 1.$$

The genus of Γ_s calculated from (2.1.1) is,

$$\gamma_s = \frac{1}{2}(kt)^{2\gamma''} a^{n^{d-1}-2} b^{n^{d-1}-2}(2ab(\gamma''-1)) + n^{d-1}(2ab-a-b)+1, k \geq 1,$$

where γ'' is given by (2.1.1.3).

Let $G_s = \Gamma/\Gamma_s$ which is a smooth quotient of Γ and $G_s \trianglerighteq G_s' \trianglerighteq G_s'' \trianglerighteq G_s''' = \{1\}$ showing that for every positive integer k ≥ 1, G_s is a solvable smooth quotient of Γ of derived length 3. The order of G_s, i.e.

$$|G_s| = \left|\Gamma/\Gamma_s\right| = \left|\Gamma/\Gamma'\right|\left|\Gamma'/\Gamma''\right|\left|\Gamma''/\Gamma^s\right|.$$

$$= d\cdot n^{d-1}\cdot(kt)^{2\gamma''}a^{n^{d-1}-1}b^{n^{d-1}-1}, k \geq 1$$

$$= (kt)^{2\gamma''}\cdot d\cdot n^{d-1}a^{\left(n^{d-1}-1\right)}b^{\left(n^{d-1}-1\right)}, k \geq 1 \qquad \text{......(2.1.1.4)}$$

where γ'' is given by (2.1.1.3). Hence the theorem.

Corollary 1 : When a = b = 1, then Γ'' is a surface group (2.1.1.2) and by Lemma (2.1.1) we get a family of solvable smooth quotient of Γ of derived length 3. In this case the order of the group is $(kt)^{2\gamma''}dn^{d-1}$ for all $k \geq 1$ and when k = 1, $\Gamma'' = \Gamma_s$ and we get a metabelian smooth quotient Γ/Γ'' of Γ of order dn^{d-1}.

Corollary 2 : If one of a and b is strictly greater than 1, i.e. either a = 1, b > 1 or a > 1, b = 1 then Γ = (d, bd, n) or Γ = (ad, d, n). The result of Chetiya [25], Chetiya and Kalita [31] comes out as special cases when d = 3, n = 2 and ℓ = 2, m = 4, n = 5 respectively. We now have,

$$\Gamma' = \left(b, \underbrace{n,\ldots,n}_{d-\text{times}}\right) \quad \text{or} \quad \Gamma' = \left(a, \underbrace{n,\ldots,n}_{d-\text{times}}\right)$$

$$\text{and} \quad \Gamma'' = \left(\gamma''; \underbrace{b,\ldots,b}_{n^{d-1}-\text{times}}\right) \quad \text{or} \quad \Gamma'' = \left(\gamma''; \underbrace{a,\ldots,a}_{n^{d-1}-\text{times}}\right).$$

In both the cases the periods of Γ'' satisfy the l.c.m. condition and we get a family of solvable smooth quotient of Γ of derived length 3(if $\gamma'' \neq 0$) and the order is obtained by putting a = 1 or b = 1 in (2.1.1.4). If d = 2, $\gamma'' = 0$ and in this case Γ''' is a normal surface subgroup of Γ'' with finite index. so by Lemma (2.1.2), Γ''' has a normal surface subgroup Γ_k containing $\Gamma^{(IV)}$ such that $[\Gamma''' : \Gamma_k] = k^{2g}$, k is any positive integer greater than or equal to 1, and g is the genus of Γ_k. Thus we get a family of solvable smooth quotient of Γ of derived length 4.

Corollary 3 : If $a \neq 1$, $b \neq 1$, $d = 2$ or $d = 3$ and $n = 2$, then $\gamma'' = 0$. Γ''' is a surface subgroup of Γ'' of finite index. In this case also by Lemma (2.1.2) we get a family of solvable smooth quotient of Γ of derived length 4.

Theorem 2.1.2. : Let $\Gamma = (\ell, m, n)$ be a Fuchsian group where ℓ, m, n are positive integers ≥ 2 such that $(\ell, m) = d_1 > 1$, $(\ell, n) = d_2 > 1$, $(m, n) = 1$ and $(d_1, d_2) = 1$ (as $(m, n) = 1$). Then for each positive integer $k > 1$, Γ admits smooth quotients of order

$$d_1 d_2 k^{2\gamma'} t^{2\gamma_k} a^{A+2\gamma_k-1} b^{A/h_1+2\gamma_k-1} c^{A/h_2+2\gamma_k-1} h_1^{d_2-A/h_1-2\gamma_k} h_2^{d_1-A/h_2-2\gamma_k}, k > 1$$

and of genus :

$$\frac{1}{2h_1^{A/h_1+2\gamma_k-1} h_2^{A/h_2+2\gamma_k-1}} \left(a^{A+2\gamma_k-1} b^{A/h_1+2\gamma_k-1} c^{A/h_2+2\gamma_k-1} t^{2\gamma_k} \right) \Bigg[2\gamma_k - 2$$

$$+ A\left(1 + \frac{1}{h_1} + \frac{1}{h_2} - \frac{1}{a} - \frac{1}{b} - \frac{1}{c} \right) \Bigg] + 1,$$

where $A = h_1^{d_2-1} h_2^{d_1-1} k^{2\gamma'}$, $h_1 = (b, k)$ and $h_2 = (c, k)$.

Proof : Let us assume that $\ell = ad_1d_2$, $m = bd_1$, $n = cd_2$ where a, b, c are prime to each other.

Let Γ be generated by elements x_1, x_2 and x_3 satisfying :

$$x_1^{ad_1d_2} = x_2^{bd_1} = x_3^{cd_2} = x_1 x_2 x_3 = 1$$

or equivalently, $x_1^{ad_1d_2} = x_2^{bd_1} = (x_1x_2)^{cd_2} = 1$.

Let u_1, u_2, u_3 be the images of x_1, x_2 and x_3 respectively under the abelianizing homomorphism from Γ onto Γ/Γ'. Then Γ/Γ' is generated by u_1, u_2 and u_3 satisfying :

$$u_1^{ad_1d_2} = u_2^{bd_1} = u_3^{cd_2} = u_1 u_2 u_3 = 1.$$

or $u_1^{ad_1d_2} = u_2^{bd_1} = (u_1u_2)^{cd_2} = 1$.

The above relations give,

$$u_1^{d_1d_2} = 1 \text{ as}$$

$$(u_1u_2)^{cd_2} = 1 \Rightarrow u_1^{cd_2}u_2^{cd_2} = 1$$

$$\Rightarrow u_1^{d_2}u_2 = 1$$

$$\Rightarrow u_2 = u_1^{-d_2}.$$

Now, $u_1^{ad_1d_2} = u_2^{bd_1} = 1$

$$\Rightarrow u_1^{ad_1d_2} = (u_1^{-d_2})^{bd_1} = 1$$

$$\Rightarrow u_1^{ad_1d_2} = u_1^{-bd_1d_2} = 1$$

Therefore $u_1^{d_1d_2} = 1$, as (a, b) = 1.

So, $\Gamma/\Gamma' \cong Z_{d_1d_2}$.

By Lemma (2.1.1),

$$\Gamma' = \left(\gamma'; a, \underbrace{b, \ldots, b}_{d_2-\text{times}}, \underbrace{c, \ldots, c}_{d_1-\text{times}} \right). \qquad \text{......(2.1.2.1)}$$

From (2.1.1) we get $\gamma' = \frac{1}{2}[(d_1 - 1)(d_2 - 1)]$.

By Lemma (2.1.4), Γ' has a subgroup Γ_k, k > 1 of finite index whose periods satisfy the l.c.m. condition.

Let $\Gamma_k^* = \{x^k, x \in \Gamma'\}$ be a subgroup of Γ', for $k \geq 2$. Let $\Gamma_k = \Gamma_k^*\Gamma''$, then by Lemma (2.1.2) Γ_k is normal in Γ' of finite index.

As $\Gamma'' \subseteq \Gamma_k \subseteq \Gamma'$, so Γ'/Γ_k is abelian.

Consider an abelianizing homomorphism :

$$\phi : \Gamma' \to \Gamma'/\Gamma_k .$$

Let $u, u_1', \dots, u_{d_2}', v_1', \dots, v_{d_1}', a_1, b_1, \dots, a_\gamma', b_\gamma'$ be the images of $x, x_1', \dots, x_{d_2}', y_1', \dots, y_{d_1}'$, $\alpha_1, \beta_1, \dots, \alpha_\gamma', \beta_\gamma'$ respectively under the above mentioned homomorphism satisfying the conditions :

$$u^a = u_1'^b = \dots = u_{d_2}'^b = v_1'^c = \dots = v_{d_1}'^c = uu_1' \dots u_{d_2}' v_1' \dots v_{d_1}' = 1$$

and $\quad u^k = u_1'^k = \dots u_{d_2}'^k = v_1'^k = \dots = v_{d_1}'^k = a_1^k = b_1^k = \dots = a_{\gamma'}^k = b_{\gamma'}^k = 1,$

where the elements commute with each other.

If (b, k) = $h_1 \geq 1$, (c, k) = $h_2 \geq 1$, then the above relations give :

$$u_1'^{h_1} = \dots = u_{d_2}'^{h_1} = v_1'^{h_2} = \dots = v_{d_1}'^{h_2} = 1$$

and the elements commute with each other.

We conclude that,

$$\Gamma'/\Gamma_k \cong \underbrace{z_{h_1} \oplus \dots \oplus z_{h_1}}_{(d_2-1)\text{ summands}} \oplus \underbrace{z_{h_2} \oplus \dots \oplus z_{h_2}}_{(d_1-1)\text{ summands}} \oplus \underbrace{z_k \oplus \dots \oplus z_k}_{2\gamma'\text{-summands}} .$$

Therefore,

$$\left| \Gamma'/\Gamma_k \right| = h_1^{d_2-1} h_2^{d_1-1} k^{2\gamma'} = A, \text{ say} \qquad \text{......(2.1.2.2.)}$$

By Lemma (2.1.1) we get,

$$\Gamma_k = \left(\gamma_k ; \underbrace{a, \dots, a}_{A\text{-times}}, \underbrace{b/h_1, \dots, b/h_1}_{A/h_1\text{-times}}, \underbrace{c/h_2, \dots, c/h_2}_{A/h_2\text{-times}} \right).$$

Since the periods of Γ_k satisfy the l.c.m. condition, therefore Γ_k' is a surface group. The genus of γ_k calculated from (2.1.1) is,

$$\gamma_k = \frac{A}{2}\left[2\gamma' - 2 + d_2\left(1 - \frac{1}{b}\right) + d_1\left(1 - \frac{1}{c}\right) - \frac{1}{h_1} - \frac{1}{h_2} + \frac{1}{b} + \frac{1}{c}\right] + 1.$$

By Lemma (2.1.3), let us construct another subgroup Γ_k^* generated by $t^{\text{th}}(t > 1)$ power of the infinite order generators of Γ_k.

Let $N = \Gamma_k^* \Gamma_k'$, then N is normal in Γ_k of finite index and Γ_k/N is abelian as

$$\Gamma_k' \subseteq \Gamma_k^* \Gamma_k' = N.$$

We now have,

$$\Gamma_k/N \cong \underbrace{Z_a \oplus \ldots \oplus Z_a}_{(A-1)\text{ summands}} \oplus \underbrace{Z_{b/h_1} \oplus \ldots \oplus Z_{b/h_1}}_{(A/h_1 - 1)\text{ summands}} \oplus \underbrace{Z_{c/h_2} \oplus \ldots \oplus Z_{c/h_2}}_{\left(A/h_2 - 1\right)\text{-summands}} \oplus \underbrace{Z_{t\ell'} \oplus \ldots \oplus Z_{t\ell'}}_{2\gamma_k\text{-summands}}$$

where $\ell' = \dfrac{abc}{h_1 h_2}$ is the l.c.m. of the periods of Γ_k.

Therefore $$\left|\Gamma_k/N\right| = a^{A-1}\left(b/h_1\right)^{A/h_1 - 1}\left(c/h_2\right)^{A/h_2 - 1}(t\ell')^{2\gamma_k}$$

$$= \frac{t^{2\gamma_k}}{h_1^{A/h_1 + 2\gamma_k - 1} h_2^{A/h_2 + 2\gamma_k - 1}}\left[a^{A + 2\gamma_k - 1} b^{A/h_1 + 2\gamma_k - 1} c^{A/h_2 + 2\gamma_k - 1}\right]$$

$$= B, \text{ say} \qquad \ldots\ldots(2.1.2.3)$$

where A is given by (2.1.2.2).

By Lemma (2.1.1) we have $N = (\gamma_n; \ldots)$.

Since N is a surface group and therefore $G = \Gamma/N$ is a smooth quotient of Γ and the genus obtained from (2.1.1) is

$$\gamma_n = \frac{B}{2}\left[2\gamma_k - 2 + A\left(1 + \frac{1}{h_1} + \frac{1}{h_2} - \frac{1}{a} - \frac{1}{b} - \frac{1}{c}\right)\right] + 1,$$

where A and B are given by (2.1.2.2) and (2.1.2.3) respectively. Therefore we have

$$\gamma_n = \frac{t^{2\gamma_k}}{2h_1^{A/h_1 + 2\gamma_k - 1} h_2^{A/h_2 + 2\gamma_k - 1}}\left(a^{A+2\gamma_k-1} b^{A/h_1 + 2\gamma_k - 1} c^{A/h_2 + 2\gamma_k - 1}\right)\left[2\gamma_k - 2\right.$$

$$\left. + A\left(1 + \frac{1}{h_1} + \frac{1}{h_2} - \frac{1}{a} - \frac{1}{b} - \frac{1}{c}\right)\right] + 1.$$

Now the order of G is,

$$|G| = \left|\Gamma/N\right| = \left|\Gamma/\Gamma'\right|\left|\Gamma'/\Gamma_k\right|\left|\Gamma_k/N\right|$$

$$= d_1 d_2 k^{2\gamma'} t^{2\gamma_k} a^{A+2\gamma_k-1} b^{A/h_1 + 2\gamma_k - 1} c^{A/h_2 + 2\gamma_k - 1} h_1^{d_2 - A/h_1 - 2\gamma_k} h_2^{d_1 - A/h_2 - 2\gamma_k}, k > 1 \cdot$$

Thus Γ admits smooth quotients, but we cannot say whether it is solvable or not. This completes the proof of the theorem.

Corollary 1. : If a = b = c = 1, then Γ admits abelian smooth quotients of order $d_1 d_2$ and of genus $\frac{1}{2}[(d_1 - 1)(d_2 - 1)]$ as well as metabelian smooth quotients of order $d_1 d_2 k^{2\gamma'}$ and of genus,

$$k^{(d_1-1)(d_2-1)}\left[\frac{1}{2}(d_1 - 1)(d_2 - 1) - 1\right] + 1, \text{ for } k \geq 1.$$

Proof : From (2.1.2.1) we have $\Gamma' = (\gamma'; \ldots)$, which shows that Γ admits smooth quotients of order $d_1 d_2$ and of genus $\frac{1}{2}[(d_1 - 1)(d_2 - 1)]$.

For the second part, we apply Lemma (2.1.2) and we define a subgroup,

$$\Gamma_k = \{x^k, x \in \Gamma' \text{ for } k \geq 1\} \text{ of } \Gamma'.$$

Let N = $\Gamma_k\Gamma''$, then N is normal subgroup of Γ of finite index.

As $\Gamma'' \subseteq \Gamma_k\Gamma'' = N$, therefore Γ/N is abelian. Consider an abelianizing homomorphism :

$$\phi : \Gamma' \to \Gamma'/N \,.$$

We get $\Gamma'/N \cong \underbrace{z_k \oplus \ldots \oplus z_k}_{2\gamma'-\text{summands}}$.

Applying Lemma (2.1.1), we have N = (γ''; ...) which shows that N contains no finite order generators. Thus N is a surface group, whose genus can be calculated from (2.1.1) and is,

$$\gamma'' = k^{(d_1-1)(d_2-1)}\left[\frac{1}{2}(d_1-1)(d_2-1)-1\right]+1, \text{ for } k \geq 1.$$

If we set $G = \Gamma/N$, then G is a smooth quotient of Γ, since N is a surface group. Also

$G' = \Gamma'/N, G'' = \Gamma''/N = \{1\}$ as $\Gamma'' \subseteq N$. Therefore $G \trianglerighteq G' \trianglerighteq G'' = \{1\}$. Hence G is metabelian.

Now the order of G i.e.,

$$|G| = \left|\Gamma/N\right|$$

$$= \left|\Gamma/\Gamma'\right|\left|\Gamma'/N\right|$$

$$= d_1d_2k^{2\gamma'}, \text{ for } k \geq 1$$

and $$\gamma' = \frac{1}{2}\left[(d_1-1)(d_2-1)\right].$$

Corollary 2 : Let a = 1 and b, c > 1.

From (2.1.2.1) we have,

$$\Gamma' = \left(\gamma'; \underbrace{b,\ldots,b}_{d_2\text{-times}}, \underbrace{c,\ldots,c}_{d_1\text{-times}} \right).$$

Therefore Γ'' is a surface group as the periods of Γ' satisfies the l.c.m. condition. Now for application of Lemma (2.1.3), let us define a subgroup,

$$\Gamma_k = \{x^{bck}, x \in \Gamma' \text{ for any positive integer } k \geq 1\}$$

of Γ', where bc is the l.c.m. of the periods of Γ'.

Let $N = \Gamma_k\Gamma''$ then N is normal in Γ'.

Further $\Gamma'' \subseteq \Gamma_k\Gamma'' = N$.

Therefore Γ'/N is abelian as it contains Γ''. Considering an abelianizing homomorphism :

$\phi : \Gamma' \to \Gamma'/N$ we get,

$$\Gamma'/N \cong \underbrace{Z_b \oplus \ldots \oplus Z_b}_{(d_2-1)\text{ summands}} \oplus \underbrace{Z_c \oplus \ldots \oplus Z_c}_{(d_1-1)\text{ summands}} \oplus \underbrace{Z_{bck} \oplus \ldots \oplus Z_{bck}}_{2\gamma'\text{-summands}}.$$

Therefore,

$$\left|\Gamma'/N\right| = b^{d_2-1}c^{d_1-1}(bck)^{2\gamma'}$$

$$= b^{d_2+2\gamma'-1}c^{d_1+2\gamma'-1}k^{2\gamma'}.$$

Applying Lemma (2.1.1) we get $N = (\gamma''; \ldots)$, which shows that N contains no finite order generators. Thus N is a surface group and the genus γ'' calculated from (2.1.1) is

$$\frac{1}{2}b^{d_2+2\gamma'-1}c^{d_1+2\gamma'-1}k^{2\gamma'}\left[d_1d_2 - \frac{d_2}{b} - \frac{d_1}{c} - 1\right] + 1; k \geq 1,$$

where $\gamma' = \frac{1}{2}(d_1-1)(d_2-1)$.

If we set $G = \Gamma/N$, then G is a smooth quotient of Γ as N is a normal surface group.

Also $G' = \Gamma'/N, G'' = \Gamma''/N = \{1\}$ as $\Gamma'' \subseteq N$.

Therefore $G \trianglerighteq G' \trianglerighteq G'' = \{1\}$ showing that G is metabelian.

The order of G, $|G| = \left|\Gamma/N\right| = \left|\Gamma/\Gamma'\right|\left|\Gamma'/N\right| = d_1 d_2 b^{d_2+2\gamma'-1} c^{d_1+2\gamma'-1} k^{2\gamma'}, k \geq 1$.

Theorem 2.1.3 : Let $\Gamma = (\ell, m, n)$ be a Fuchsian group where ℓ, m, n are positive integers ≥ 2 such that $(\ell, m, n) = r \geq 1$ and $(\ell, m) = rd_1$, $(\ell, n) = rd_2$, $(m, n) = rd_3$, for some non-negative integers d_1, d_2 and d_3, where d_1, d_2, d_3 are pairwise prime to each other. Then Γ admits a metabelian smooth quotients of order,

$$r^2 d_1^2 d_2 d_3 k^{2\gamma'} a^{rd_1d_3+2\gamma'-1} b^{rd_1d_2+2\gamma'-1} c^{rd_1^2+2\gamma'-1}$$

and of genus,

$$\frac{1}{2}\left(a^{rd_1d_3+2\gamma'-1} b^{rd_1d_2+2\gamma'-1} c^{rd_1^2+2\gamma'-1} k^{2\gamma'}\right)\left[2\gamma' - 2 \right.$$

$$\left. + rd_1\left(d_1 + d_2 + d_3 - \frac{d_3}{a} - \frac{d_2}{b} - \frac{d_1}{c}\right)\right] + 1, k \geq 1.$$

Proof : Let us take $\ell = ard_1d_2$, $m = brd_1d_3$ and $n = crd_2d_3$.

Let Γ be generated by elements x_1, x_2 and x_3 satisfying :

$$x_1^{ard_1d_2} = x_2^{brd_1d_3} = x_3^{crd_2d_3} = x_1x_2x_3 = 1,$$

or equivalently,

$$x_1^{ard_1d_2} = x_2^{brd_1d_3} = (x_1x_2)^{crd_2d_3} = 1.$$

Let u_1, u_2, u_3 be the images of x_1, x_2 and x_3 respectively under the abelianizing homomorphism from Γ onto Γ/Γ'. Then Γ/Γ' is generated by u_1, u_2 and u_3 satisfying :

$$u_1^{ard_1d_2} = u_2^{brd_1d_3} = u_3^{crd_2d_3} = u_1u_2u_3 = 1,$$

or $$u_1^{ard_1d_2} = u_2^{brd_1d_3} = (u_1u_2)^{crd_2d_3} = 1.$$

Now $$(u_1u_2)^{crd_2d_3} = 1$$

$$\Rightarrow u_1^{crd_2d_3}u_2^{crd_2d_3} = 1$$

$$\Rightarrow u_1^{rd_2} = u_2^{-rd_3}.$$

Now $$u_2^{brd_1d_3} = 1$$

$$\Rightarrow (u_2^{-rd_3})^{-bd_1} = 1$$

$$\Rightarrow u_1^{rbd_1d_2} = 1$$

$$\Rightarrow u_1^{rd_1d_2} = 1.$$

Also, $$u_1^{ard_1d_2} = 1$$

$$\Rightarrow \left(u_1^{-rd_2}\right)^{(-ad_1)} = 1$$

$$\Rightarrow \left(u_2^{rd_3}\right)^{(-ad_1)} = 1$$

$$\Rightarrow u_2^{ard_1d_3} = 1$$

$$\Rightarrow u_2^{rd_1d_3} = 1.$$

So, $\Gamma/\Gamma' \cong Z_{rd_1d_2} \oplus Z_{rd_1d_3}$.

By Lemma (2.1.1) we get

$$\Gamma' = \left(\gamma';\ \underbrace{a,\ldots,a}_{rd_1d_3-\text{times}},\ \underbrace{b,\ldots b}_{rd_1d_2-\text{times}},\ \underbrace{c,\ldots,c}_{rd_1^2-\text{times}}\right) \qquad \ldots\ldots(2.1.3.1)$$

where γ' can be calculated from (2.1.1) and is

$$\gamma' = \frac{rd_1}{2}[rd_1d_2d_3 - d_1 - d_2 - d_3] + 1. \qquad ......(2.1.3.2)$$

Obviously $\gamma' \neq 0$.

Since the periods of Γ' satisfy the l.c.m. condition, therefore Γ'' is a surface group.

Let us define a subgroup,

$$\Gamma_k = \{ x^{\ell_1 k} \ ; \ x \in \Gamma' \text{ for any positive integer } k \geq 1 \}$$

of Γ', where ℓ_1 = abc is the l.c.m. of the periods of Γ'.

Let $N = \Gamma_k\Gamma''$ and N is normal in Γ'.

Further $\Gamma'' \subseteq \Gamma_k\Gamma'' = N$.

Therefore Γ'/N is abelian (as N contains Γ'').

Considering an abelianizing homomorphism :

$$\phi : \Gamma' \rightarrow \Gamma'/N$$

we have $\Gamma'/N \cong \underbrace{z_a \oplus \ldots \oplus z_a}_{(rd_1d_3-1)\text{ summands}} \oplus \underbrace{z_b \oplus \ldots \oplus z_b}_{(rd_1d_2-1)\text{ summands}} \oplus \underbrace{z_c \oplus \ldots \oplus z_c}_{(rd_1^2-1)\text{ summands}} \oplus \underbrace{z_{k\ell_1} \oplus \ldots \oplus z_{k\ell_1}}_{2\gamma'\text{-summands}}$.

Therefore,

$$\left|\Gamma'/N\right| = a^{rd_1d_3-1}b^{rd_1d_2-1}c^{rd_1^2-1}(k\ell_1)^{2\gamma'}.$$

$$= a^{rd_1d_3+2\gamma'-1}b^{rd_1d_2+2\gamma'-1}c^{rd_1^2+2\gamma'-1}k^{2\gamma'}, k \geq 1 \cdot$$

$$= A, \text{ say.}$$

Applying Lemma (2.1.1) we have,

$$N = (\gamma''; \ldots),$$

which shows that N contains no finite order generators. Hence N is a surface group and

$$\gamma'' = \frac{1}{2}\left(a^{rd_1d_3+2\gamma'-1}b^{rd_1d_2+2\gamma'-1}c^{rd_1^2+2\gamma'-1}k^{2\gamma'}\right)\left[2\gamma'-2\right.$$

$$\left. +rd_1\left(d_1+d_2+d_3-\frac{d_3}{a}-\frac{d_2}{b}-\frac{d_1}{c}\right)\right]+1 \quad \text{......(2.1.3.3)}$$

where γ' is given by (2.1.3.2)

If we set $G = \Gamma/N$, then G is a smooth quotient of Γ, since N is a surface group.

Also $G' = \Gamma'/N, G'' = \Gamma''/N = \{1\}$ as $\Gamma'' \subseteq \Gamma_k\Gamma'' = N$.

Therefore $G \trianglerighteq G' \trianglerighteq G'' = \{1\}$.

Hence G is metabelian. Now the order of G i.e.,

$$|G| = \left|\Gamma/N\right|$$

$$= \left|\Gamma/\Gamma'\right|\left|\Gamma'/N\right|$$

$$= r^2d_1^2d_2d_3A.$$

Therefore,

$$|G| = r^2d_1^2d_2d_3a^{rd_1d_3+2\gamma'-1}b^{rd_1d_2+2\gamma'-1}c^{rd_1^2+2\gamma'-1}k^{2\gamma'}, k \geq 1.$$

Hence the theorem.

Theorem 2.1.3 gives the following corollary :

Corollary 1 : If a = b = c = 1, then Γ admits a family of abelian smooth quotients of order $r^2d_1^2d_2d_3$ and of genus,

$$\frac{rd_1}{2}(rd_1d_2d_3-d_1-d_2-d_3)+1$$

as well as an infinite family of metabelian smooth quotients of order $r^2d_1^2d_2d_3k^{2\gamma'}$

and of genus $k^{2\gamma'}(\gamma'-1)+1$ for each k > 1, where $\gamma'=\frac{rd_1}{2}(rd_1d_2d_3-d_1-d_2-d_3)+1$.

The following cases were studied by P. Bhattacharjee in her doctoral thesis which will come out as special cases in our study.

I. $\Gamma = (2, m, n)$ where m, n are odd positive integers such that $2 < m \leq n$ and if (i) $m = n$ then $\Gamma = (2, m, m)$ has a solvable smooth quotient of derived length ≤ 3. (ii) $m \neq n$, then $\Gamma = (2, m, n)$ has a solvable smooth quotient of derived length ≤ 4.

II. $\Gamma = (2, m, n)$ where m, n are even positive integers $2 < m \leq n$, $(m, n) = d \geq 2$ then if (i) $m = n$, Γ has a solvable smooth quotient of derived length ≤ 2. (ii) $m \neq n$, then Γ has a solvable smooth quotient of derived length ≤ 3.

III. $\Gamma = (2, m, n)$ where m odd and n even, $(m, n) = d$, $2 < m < n$ and if (i) $d = 1$ then Γ has a solvable smooth quotient of derived length ≤ 4. (ii) if $d > 1$ then Γ has a smooth quotient.

2.2 Conclusion

From the above theorems we have come to the conclusion that for any admissible triangle group we can prove the existence of finite solvable smooth quotients of the Fuchsian triangle group whose derived length is less than or equal to 4.

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Chapter 3 : *Solution to the minimum genus problem of a class of Zs-metacyclic groups of automorphisms of compact Riemann surfaces*

CHAPTER - 3

SOLUTION TO THE MINIMUM GENUS PROBLEM OF A CLASS OF ZS-METACYCLIC GROUPS OF AUTOMORPHISMS OF COMPACT RIEMANN SURFACES

In chapter 1, we mentioned about the famous minimum genus problem which is to find the minimum possible value of the genus $g(\geq 2)$ of a compact Riemann surface admitting a finite group G as its automorphism group. This problem in its generality seems to be too difficult to solve. Harvey [70] and Maclachlan [102] solved this problem for cyclic and non-cyclic abelian groups respectively, while Glover and Sjerve [57] solved the problem for PSL(2, p). The same problem for a class of non-abelian groups have been partially solved by Chetiya and Patra [35], Chetiya, Dutta and Patra [26], and in the doctoral thesis of P. Ahmed [9] where the class of K-metacyclic groups, Zs-metacyclic groups of order pq where p and q are primes and quaternion groups were considered respectively. Chetiya and Ahmed [8] studied the quaternion groups of order 4m belonging to the family of metacyclic groups which are two generator groups.

Also Chetiya and Chutiya studied an important subclass of metabelian groups i.e., a dihedral group D_{2n} of order 2n for every positive integer n.

As a first step towards solving the minimum genus problem for a general Zs-metacyclic group, we consider a Zs-metacyclic group M of order $4p^2$ where p is an odd prime.

A group M is called a metacyclic group if there is a normal subgroup N of M such that both N and M/N are cyclic. Thus a metacyclic group is a metabelian group which has an invariant series $M \trianglerighteq N \trianglerighteq \{1\}$ of length two. A Z-metacyclic group or a Zassenhaus metacyclic group [139] is a metacyclic group of special type. In a Z-metacyclic group M, the derived group M' and M/M' are cyclic. Every Z-metacyclic group of order 'ℓn' can be generated by two elements a and b with defining relations :

$$a^{\ell} = b^{n} = 1, \; b^{-1}ab = a^{r}$$

where $r^n \equiv 1 \pmod{\ell}$, $(r - 1, \ell) = 1$

and $(r - 1, \ell)$ denoting the h.c.f. of $r - 1$ and ℓ.

A Z-metacyclic group M is called a Zs-metacyclic group if all its sylow subgroups are cyclic. The most important case of a Zs-metacyclic group is,

$$a^p = b^{p-1} = 1,\ b^{-1}ab = a^r;\ (r - 1, p) = 1,$$

where p is any odd prime and r is a primitive root (modulo p). A Zs-metacyclic group with the above defining relations is called a K-metacyclic group.

In this chapter we consider a particular Zs-metacyclic group M of order $4p^2$, p an odd prime ≥ 3 having a presentation :

$$\left\langle a, b : a^{p^2} = b^4 = 1, b^{-1}ab = a^{-1} \right\rangle. \qquad ......(3.1)$$

It is known that any finite group and so also M acts as a group of automorphisms of a compact Riemann surfaces of genus $g \geq 2$. Again the same group is representable as a group of automorphisms of compact Riemann surfaces of different genera. The objective of this chapter is to find the minimum such genus of a Riemann surface admitting M as an automorphism group.

Throughout the rest of this chapter we shall denote by M, the Zs-metacyclic group of order $4p^2$, p an odd prime ≥ 3 with above presentation (3.1).

We know that a Fuchsian group Γ with presentation :

$$\left\langle x_1, \ldots, x_s, \alpha_1, \beta_1, \ldots, \alpha_\gamma, \beta_\gamma : x_1^{m_1} = \ldots = x_s^{m_s} = \prod_{i=1}^{s} x_i \prod_{i=1}^{\gamma} [\alpha_i, \beta_i] = 1 \right\rangle ......(3.2)$$

where $[\alpha_i, \beta_i] = \alpha_i^{-1}\beta_i^{-1}\alpha_i\beta_i$ with

$$\delta(\Gamma) = 2\gamma - 2 + \sum_{i=1}^{s}\left(1 - \frac{1}{m_i}\right) > 0 \qquad ......(3.3)$$

is said to have the ***signature*** $(\gamma; m_1, \ldots, m_s)$. The integers $m_i (\geq 2)$ are called the ***periods*** and γ the ***genus*** of Γ. If $\gamma = 0$ then the signature of Γ will be written as $(m_1, \ldots, m_s)$ when $\gamma = 0$ and $s = 3$ the group is called a Fuchsian triangle group. In view of (3.3) when

$\gamma = 0$ the following sets of values for the periods of Γ are untenable :

(i) $\{m, n\}$ (ii) $\{2, 2, m\}$ (iii) $\{2, 3, 3\}$

(iv) $\{2, 3, 4\}$ (v) $\{2, 3, 5\}$ (vi) $\{2, 3, 6\}$

(vii) $\{2, 2, 2, 2\}$ (viii) $\{3, 3, 3\}$ (ix) $\{2, 4, 4\}$.

If Γ has a presentation (3.2) then by Riemann Hurwitz formula,

$$\frac{2(g-1)}{|G|} = 2(\gamma-1) + \sum_{i=1}^{s}\left(1-\frac{1}{m_i}\right), \qquad \text{.......(3.4)}$$

Minimum of g will be obtained when the right side of (3.4) is minimum.

In the next section we establish a necessary and sufficient condition for the existence of a smooth epimorphism from a Fuchsian group Γ onto M and in section 3.2 we obtain the minimum genus g of the compact Riemann surface having M as its automorphism group, and the corresponding signature of the Fuchsian group Γ having M as a smooth quotient.

3.1 Existence of smooth epimorphism

Let Γ be a Fuchsian group and M a Zs-metacyclic group having presentation (3.1). Now any element of M is of the type $b^j a^i$ where $0 \le j < 4$, $0 \le i < p^2$ [69]. We list below the orders of the elements of M for different values of j and i :

Elements of the form	**Orders**
b^2	2
ba^i, b^3a^i; $0 \le i < p^2$	4
a^{pt}, $1 \le t < p$	p
a^i, $(i, p^2) = 1$	p^2
b^2a^{pt}, $1 \le t < p$	2p
b^2a^k, $(k, p^2) = 1$	$2p^2$.

Let $\phi : \Gamma \to M$ be a smooth epimorphism. Then

$$\prod_{i=1}^{s}\phi(x_i)\prod_{i=1}^{\gamma}[\phi(\alpha_i),\phi(\beta_i)] = 1$$

i.e. $$\prod_{i=1}^{s}\phi(x_i) = \left\{\prod_{i=1}^{\gamma}[\phi(\alpha_i),\phi(\beta_i)]\right\}^{-1} \in M', \qquad ....(3.1.1)$$

where M' is the derived group of M. Now any commutator of M is of the form :

$$\begin{aligned}[b^ia^j, b^sa^k] &= a^{-j}b^{-i}a^{-k}b^{-s}b^ia^jb^sa^k \\ &= a^{-j}a^{-k(-1)^i}a^{j(-1)^s}a^k \\ &= a^{-j+k(-1)^{i+1}+j(-1)^s+k} \\ &= a^{k\{(-1)^{i+1}+1\}+j\{(-1)^s-1\}} \ . \qquad ......(3.1.2)\end{aligned}$$

Now for different values of i and s we get the commutator in the following forms :

i and s both even	***i and s both odd***	***i even s odd***	***i odd s even***
1	$a^{2(k-j)}$	a^{-2j}	a^{2k}

We have seen that the commutator $[b^ia^j, b^sa^k]$ is any power of a or identity i.e.

i.e. $[b^ia^j, b^sa^k] = a^{2\ell}$ say, where $(\ell, p^2) = 1$. ....(3.1.3)

Therefore M' is generated by a i.e., $M' = \langle a \rangle$.

We now list below the finite product of any finite power of the above elements. In our main theorem we shall prove that t_4 which is the number of occurences of the element ba^i or b^3a^i is always even. So in the product we shall consider only even powers of ba^i or b^3a^i(where one is the inverse of the other).

I. (i) $(b^2)^{t_2} = 1 \in M'$ for even t_2.

$= b^2 \notin M'$ for odd t_2.

(ii) Let $t_4 = t_4' + t_4''$.

Case (i) : t_4' and t_4'' are both odd.

Let $t_4' = 2m + 1$, $t_4'' = 2n + 1$; m, n ≥ 0.

Therefore $(ba^i)^{t_4'}(b^3a^i)^{t_4''} = 1 \in M'$, when m, n are of same parity i.e. m and n both are even or both odd.

$= b^2 \notin M'$, when m, n are of opposite parity.

Case (ii) : t_4' and t_4'' are both even.

Then $(ba^i)^{t_4'}(b^3a^i)^{t_4''} = 1 \in M'$.

(iii) $(a^{pt})^{t_p} \in M', 1 \le t < p$.

(iv) $(a^i)^{t_{p^2}} \in M', (i, p^2) = 1$.

(v) $(b^2a^{pt})^{t_{2p}} = a^{ptt_{2p}} \in M'$ for t_{2p} even.

$= b^2a^{ptt_{2p}} \notin M'$ for t_{2p} odd.

(vi) $(b^2a^k)^{t_{2p^2}} = a^{kt_{2p^2}} \in M'$ for t_{2p^2} even.

$= b^2a^{kt_{2p^2}} \notin M'$ for t_{2p^2} odd.

We now find the finite product of the power of the elements b^2, (ba^i or b^3a^i), b^2a^{pt}, b^2a^k; where $(k, p^2) = 1$ and $(i, p^2) = 1$ so that the result belongs to M'.

We exclude the elements a^{pt} and a^i from the product, because the product of any power of this two elements always belongs to $M' = <a>$.

II. Let us take the product,

$$(b^2)^{t_2}(ba^i)^{t_4'}(b^3a^i)^{t_4''}(b^2a^{pt})^{t_{2p}}(b^2a^k)^{t_{2p^2}}; [t_4 = t_4' + t_4'']$$

and we already have

$$(ba^i)^{t_4'}(b^3a^i)^{t_4''} = 1,$$

when both t_4' and t_4'' are even. So we consider the product :

$$(b^2)^{t_2}(b^2a^{pt})^{t_{2p}}(b^2a^k)^{t_{2p^2}} \qquad \text{.......(A)}$$

for t_4' and t_4'' both even; and for t_4' and t_4'' both odd ($t_4' = 2m + 1$, $t_4'' = 2n + 1$; $m\ n \ge 0$). We consider the following product :

$$(b^2)^{t_2}(ba^i)^{t_4'}(b^3a^i)^{t_4''}(b^2a^{pt})^{t_{2p}}(b^2a^k)^{t_{2p^2}}. \qquad \text{......(B)}$$

Now ,

(a) When t_2, t_{2p}, t_{2p^2} all are even then the above product,

- (i) (A) belongs to M′.
- (ii) (B) belongs to M′ if m and n are of same parity.
- (iii) (B) does not belong to M′ if m and n are of opposite parity.

(b) When t_2, t_{2p}, t_{2p^2} all are odd then the above product

- (i) (A) doesnot belong to M′.
- (ii) (B) belongs to M′ if m and n are of opposite parity.
- (iii) (B) does not belong to M′ if m and n are of same parity.

(c) When one of the t_2, t_{2p}, t_{2p^2} is even and the remaining two are odd then

- (i) (A) belongs to M′.
- (ii) (B) belongs to M′ if m and n are of same parity.
- (iii) (B) does not belong to M′ if m and n are of opposite parity.

(d) When one of the t_2, t_{2p}, t_{2p^2} is odd and the remaining two are even then

- (i) (A) does not belong to M′.
- (ii) (B) belongs to M′ if m and n are of opposite parity.
- (iii) (B) does not belong to M′ if m and n are of same parity.

We are now in a position to establish the existence of smooth epimorphisms from a Fuchsian group Γ to the Zs-metacyclic group M of order $4p^2$, where $p \geq 3$ is an odd prime.

Theorem 3.1.1 : Let $\Gamma = (\gamma; m_1, \ldots, m_s)$, $s \geq 0$ be a Fuchsian group of genus γ, then there exists a smooth epimorphism from Γ to a Zs-metacyclic group M of order $4p^2$ where,

$$M = \left\langle a, b : a^{p^2} = b^4 = 1, b^{-1}ab = a^{-1}, p \geq 3 \text{ is an odd prime} \right\rangle,$$

if and only if the periods and genus of Γ satisfy the following conditions :

(1) If Γ is a surface group then $\gamma \geq 2$.

(2) If Γ is not a surface group then the periods of Γ must take the values from the set $\{2, 4, p, p^2, 2p, 2p^2\}$.

(3) If t_2, t_4, t_p, t_{p^2}, t_{2p} and t_{2p^2} denote respectively the number of periods 2, 4, p, p^2, 2p and $2p^2$ then

(A) t_4 must be even.

(B) If $s = 1$ then (i) $\Gamma = (\gamma; p)$; $\gamma \geq 2$,

or (ii) $\Gamma = (\gamma; p^2)$; $\gamma \geq 1$.

(C) If $s = 2$, then Γ has the following signatures :

(i) $(\gamma; 2, 2)$; $\gamma \geq 2$.

(ii) $(\gamma; 2, 2p)$; $\gamma \geq 2$.

(iii) $(\gamma; 2, 2p^2)$; $\gamma \geq 1$.

(iv) $(\gamma; 4, 4)$; $\gamma \geq 1$.

(v) $(\gamma; p, p)$; $\gamma \geq 2$.

(vi) $(\gamma; p, p^2)$; $\gamma \geq 1$.

(vii) $(\gamma; p^2, p^2)$; $\gamma \geq 1$.

(viii) $(\gamma; 2p, 2p)$; $\gamma \geq 2$.

(ix) $(\gamma; 2p, 2p^2)$; $\gamma \geq 1$.

(x) $(\gamma; 2p^2, 2p^2)$; $\gamma \geq 1$.

(D) If $s \geq 3$ and $t_4 = 0$ then $\gamma \geq 1$.

(a) If t_4, t_{p^2}, t_{2p^2} are simultaneously zero then $\gamma \geq 2$.

(b) If 'i' is the only period of Γ, then t_i cannot take odd values for the following values of i :

$i = 2$, $2p$ and $2p^2$.

(c) (i) If i, j, k ($i \neq j \neq k$) are the three periods of Γ then t_i, t_j, t_k cannot take odd values simultaneously for the following values of i, j, k :

$i = 2$, $j = 2p$, $k = 2p^2$.

(ii) If there are two periods of Γ instead of three, taking values from the set $\{2, 2p, 2p^2\}$ then the number of them will be both even or both odd.

(E) If $s \geq 3$, $t_4 = 2$ and (i) t_{p^2}, t_{2p^2} are simultaneously zero then $\gamma \geq 1$ otherwise $\gamma \geq 0$. (ii) The conditions mentioned in 3(D) (b) (c)(i)(ii) will also prevail in this case $[t_4 = t_4' + t_4'' \neq 0]$. In this case γ may be greater than or equal to zero. But when t_{p^2} and t_{2p^2} both are simultaneously zero then $\gamma \geq 1$.

Proof :

Necessity : (1) Let Γ be a surface group and

$$\phi : \Gamma \to M$$

be a smooth epimorphism. Then it is obvious that $\gamma \geq 2$ as

$$\delta(\Gamma) = 2\gamma - 2 + \sum_{i=1}^{s}\left(1 - \frac{1}{m_i}\right) > 0.$$

2. Let Γ is not a surface group and let $\phi : \Gamma \to M$ be a smooth epimorphism. Now any element of M other than the identity has order 2 or 4 or p or p^2 or 2p or $2p^2$. $\phi : \Gamma \to M$ is a smooth epimorphism if and only if ϕ preserves the periods of Γ [70]. Thus it follows that a period of Γ, if there is one, is an element of the set $\{2, 4, p, p^2, 2p, 2p^2\}$.

3(A). Let t_2, t_4, t_p, t_{p^2}, t_{2p} and t_{2p^2} denote respectively the number of periods 2, 4, p, p^2, 2p and $2p^2$.

Let $\phi : \Gamma \to M$ be a smooth epimorphism then the kernel K of ϕ is a surface group of genus g(say). But then from the relation (3.4) we get,

$$g = 1 + 4p^2(\gamma - 1) + t_2p^2 + \frac{3}{2}p^2t_4 + 2p(p-1)t_p + 2(p^2-1)t_{p^2}$$
$$+ p(2p-1)t_{2p} + (2p^2-1)t_{2p^2}.$$

Since 'g' is an integer therefore t_4 must be even.

3(B). Let $\phi : \Gamma \to M$ be a smooth epimorphism and $\Gamma = (\gamma; m_1)$. In view of

$$\delta(\Gamma) = 2\gamma - 2 + \left(1 - \frac{1}{m_1}\right) > 0$$

we have $\gamma \geq 1$.

Let x_1 be the element of order m_1. As $\phi(x_1) \in M' = \langle a \rangle$, so $m_1 = p$ or p^2 since M' contains the elements of order either p or p^2. Let us consider the case when $m_1 = p$. If possible let $\gamma = 1$. As ϕ preserves the periods of Γ so $\phi(x_1) = a^{pt}$, $1 \leq t < p$ and ϕ is an epimorphism gives :

$$\phi(\alpha_1) = b^i a^j;\ 0 \leq i < 4,\ 0 \leq j < p^2$$

$$\phi(\beta_1) = b^s a^k;\ 0 \leq s < 4,\ 0 \leq k < p^2$$

where at least one of i and s must be odd and at least one of j and k be such that

$$(j, p) = 1 \text{ and } (k, p) = 1.$$

Without loss of generality let i be odd and $(k, p) = 1$.

For ϕ to be an epimorphism we must have

$$\phi(x_1)[\phi(\alpha_1), \phi(\beta_1)] = 1. \qquad \text{......(3.1.1.1)}$$

Now $[\phi(\alpha_1), \phi(\beta_1)] = a^{2\ell}$, $(\ell, p^2) = 1$. [from 3.1.3]

So (3.1.1.1) gives $pt + 2\ell \equiv 0 \pmod{p^2}$

i.e. $pt + 2\ell \equiv 0 \pmod{p}$

The above congruence does not have any solution as $(\ell, p) = 1$.

When i and s both are odd then we have [from 3.1.2]

$$pt + 2(k - j) \equiv 0 \pmod{p^2}$$

i.e. $2(k - j) = p^2t' - pt = p(pt' - t)$, $t' \geq 1$

i.e. $k - j \equiv 0 \pmod{p}$

i.e. $k \equiv j \pmod{p}$

i.e. $k - j = pm$ say, $m \geq 1$.

Therefore we can contruct $\phi : \Gamma \to M$ as follows :

(a) $\phi(x_1) = a^{-2pm}$ (b) $\phi(x_1) = a^{-2pm}$ (c) $\phi(x_1) = a^{-2pm}$

$\phi(\alpha_1) = ba^j$ $\phi(\alpha_1) = b^3a^j$ $\phi(\alpha_1) = ba^j$

$\phi(\beta_1) = ba^k$ $\phi(\beta_1) = b^3a^k$ $\phi(\beta_1) = b^3a^k$.

Now from (3.1.3) we have,

$$[\phi(\alpha_1), \phi(\beta_1)] = a^{2\ell}, \text{ where } (\ell, p^2) = 1.$$

Therefore we have seen that no finite product of the above mentioned elements in each cases (a), (b) and (c) gives together a and b the generators of M. Therefore ϕ is not an epimorphism, which is a contradiction. Hence $\gamma \geq 2$ when $\Gamma = (\gamma; p)$.

3(C). Let $s = 2$ i.e., $\Gamma = (\gamma; m_1, m_2)$.

In view of $\delta(\Gamma) = 2\gamma - 2 + \left(1 - \frac{1}{m_1}\right) + \left(1 - \frac{1}{m_2}\right) > 0$, we get $\gamma \geq 1$.

Let Γ be generated by $x_1, x_2, \alpha_1, \beta_1, \ldots, \alpha_\gamma, \beta_\gamma$ where x_1 is of order m_1, x_2 is of order m_2 and α_j, β_j; $j = 1, 2, \ldots, \gamma$ are of infinite order.

Let $\phi : \Gamma \to M$ be a smooth epimorphism. Then we have,

$$\phi(x_1)\phi(x_2)\prod_{j=1}^{\gamma}[\phi(\alpha_j), \phi(\beta_j)] = 1$$

or
$$\phi(x_1)\phi(x_2) = \left\{\prod_{j=1}^{\gamma}[\phi(\alpha_j), \phi(\beta_j)]\right\}^{-1} \in M' = \langle a \rangle.$$

So $\phi(x_1)\phi(x_2) = 1$ or a power of a.

We now give a list of the product of any two elements of M (elements may be repeated). If we multiply two elements of order 4 then their product belongs to M' only when one element is taken as the inverse of the other because otherwise say x is an element of M of order 4 then the order of x^2 is 2 and so $x^2 \notin M'$ as M' does not contain any second order element.

(i) If the order of one element is 2 and other is p then their product does not belong to M'.

(ii) If the order of one element is a divisor of 4 and the other a divisor of p^2, then their product does not belong to M' as $(4, p^2) = 1$.

(iii) If the order of one element is 2 and other is the order either $2p$ or $2p^2$ then their product belong to M'.

(iv) If the order of one element is $2p$ (and $2p^2$) and other is p or p^2 then their product does not belong to M'.

(v) Product of elements of order $2p$ and $2p^2$ or $2p$ and $2p$ or $2p^2$ and $2p^2$ belongs to M'.

From the above list we can conclude that if Γ is a Fuchsian group having two periods then for the existence of a smooth epimorphism from Γ onto M, Γ must have the following signatures :

(i) $\Gamma = (\gamma; 2, 2)$ (ii) $\Gamma = (\gamma; 2, 2p)$

(iii) $\Gamma = (\gamma; 2, 2p^2)$ (iv) $\Gamma = (\gamma; 4, 4)$

(v) $\Gamma = (\gamma; p, p)$ (vi) $\Gamma = (\gamma; p, p^2)$

(vii) $\Gamma = (\gamma; p^2, p^2)$ (viii) $\Gamma = (\gamma; 2p, 2p)$

(ix) $\Gamma = (\gamma; 2p, 2p^2)$ (x) $\Gamma = (\gamma; 2p^2, 2p^2)$.

For all the cases γ should be greater than or equal to 1, but in the following cases $\gamma \geq 2$:

(a) $(\gamma; 2, 2)$ (b) $(\gamma; 2, 2p)$

(c) $(\gamma; p, p)$ (d) $(\gamma; 2p, 2p)$.

If possible, let $\gamma = 1$ for each of the above cases.

Let $\phi : \Gamma \to M$ be constructed as follows :

for case (a) $\phi(x_1) = b^2$

$\phi(x_2) = b^{-2}(\text{or } b^2)$

$\phi(\alpha_1) = b^i a^j; (0 \leq i < 4, 0 \leq j < p^2)$

$\phi(\beta_1) = b^s a^k; (0 \leq s < 4, 0 \leq k < p^2)$

for case (b) $\phi(x_1) = b^2$

$\phi(x_2) = b^2 a^{pt}; 1 \leq t < p$

$\phi(\alpha_1) = b^i a^j; (0 \leq i < 4, 0 \leq j < p^2)$

$\phi(\beta_1) = b^s a^k;(0 \leq s < 4, 0 \leq k < p^2)$

for case (c) $\phi(x_1) = a^{pt}$

$\phi(x_2) = a^{-pt}$

$\phi(\alpha_1) = b^i a^j; (0 \leq i < 4, 0 \leq j < p^2)$

$\phi(\beta_1) = b^s a^k;(0 \leq s < 4, 0 \leq k < p^2)$

for case (d) $\phi(x_1) = b^2 a^{pt}; 1 \leq t < p$

$\phi(x_2) = (b^2 a^{pt})^{-1}; 1 \leq t < p$

$\phi(\alpha_1) = b^i a^j; (0 \leq i < 4, 0 \leq j < p^2)$

$\phi(\beta_1) = b^s a^k;(0 \leq s < 4, 0 \leq k < p^2)$.

By the agrument given in the proof 3(B) we don't get the generators of M and hence ϕ will not be an epimorphism in this case. Which will be a contradiction. Hence $\gamma \geq 2$ for the above four cases.

3(D). If possible, let $\gamma = 0$ for $t_4 = 0$. Let $x_1, x_2, \ldots, x_s$ be the generators of Γ having orders 2, p, p^2, 2p, $2p^2$ only (since $t_4 = 0$). Now if ϕ is a smooth epimorphism then $\phi(x_1), \phi(x_2), \ldots, \phi(x_s)$ must generate M. $\phi(x_i)$ has order either 2 or p or p^2 or 2p or $2p^2$ for each i. Now from the list II mentioned in section 3.1 we see that no finite product of the finite power of the above ordered elements gives b. Therefore ϕ is not an epimorphism, which is a contradiction. Hence to get the whole group M atleast one pair of infinite ordered elements must be mapped to b and so $\gamma \neq 0$ i.e., $\gamma \geq 1$.

3.D.(a) : Let $\phi : \Gamma \rightarrow M$ be a smooth epimorphism where

$$\Gamma = \left(\gamma; \underbrace{2,\ldots,2}_{t_2\text{-times}}, \underbrace{p,\ldots,p}_{t_p\text{-times}}, \underbrace{2p,\ldots,2p}_{t_{2p}\text{-times}} \right).$$

If possible, let $\gamma = 1$ and $x_1, x_2, .., x_{t_2}; z_1, z_2, \ldots, z_{t_p}; v_1, v_2, \ldots, v_{t_{2p}}$ be the generators of orders 2, p and 2p respectively.

i.e., $\phi(x_i) = b^2, i = 1, 2, \ldots, t_2$.

$\phi(z_j) = a^{pt_j}$; $1 \leq t_j < p$ and $j = 1, 2, \ldots, t_p$.

$\phi(v_k) = b^2a^{pt_k}$; $1 \leq t_k < p$ and $k = 1, 2, \ldots, t_{2p}$.

Since no finite product of the finite powers of $b^2, a^{pt_j}, b^2a^{pt_k}$ will give us a and b the generators of M, so ϕ will be an epimorphism if we map :

$\phi(\alpha_1) = b$ and $\phi(\beta_1) = a^{-y}$; $(y, p) = 1$.

Now since ϕ is a homomorphism we must have :

$$\prod_{i=1}^{t_2}\phi(x_i)\prod_{j=1}^{t_p}\phi(z_j)\prod_{k=1}^{t_{2p}}\phi(v_k)[\phi(\alpha_1),\phi(\beta_1)] = 1$$

which gives, $$(b^2)^{t_2}a^{\sum_{j=1}^{t_p}pt_j}\sum_{k=1}^{t_{2p}}b^2a^{pt_k} = [b,a^{-y}]^{-1} \in M'.$$

i.e. $$a^{\sum_{j=1}^{t_p}pt_j+\sum_{k=1}^{t_{2p}}pt_k} = a^{2y}$$

i.e. $$a^{p\left[\sum_{j=1}^{t_p}t_j+\sum_{k=1}^{t_{2p}}t_k\right]} = a^{2y}.$$

[because t_2 and t_{2p} both must be either even or odd[see the proof 3.D.(c)(ii)] and using I(i) and (v) of the list mentioned in section 3.1]

i.e. $2y - pk' \equiv 0 \pmod{p^2}$ where $k' = \sum_{j=1}^{t_p}t_j + \sum_{k=1}^{t_{2p}}t_k$.

i.e. $2y - pk' = p^2t$, $t \geq 1$

i.e. $2y = p(pt + k')$, $t \geq 1$

which is impossible as $(y, p) = 1$.

Hence for ϕ to be a homomorphism we must have $\gamma \geq 2$.

3. D. (b) : If possible, let t_i be odd for the following values of i :

(i) $i = 2$ (ii) $i = 2p$ (iii) $i = 2p^2$.

Let t_2, t_{2p} and t_{2p^2} denote respectively the number of generators of order 2, 2p and $2p^2$. Now the product of the odd powers of the elements of order :

(i) 2 does not belong to M' [using I(i) of the list given in section 3.1]

(ii) 2p does not belong to M' [using I(v) of the list given in section 3.1]

(iii) $2p^2$ does not belong to M' [using I(vi) of the list given in section 3.1]

which is a contradiction. Hence t_i cannot take odd values for $i = 2$, $2p$ and $2p^2$.

3. D. (c)(i) : If possible, let t_i, t_j, t_k takes odd values simultaneously for the following values of i, j and k i.e. $i = 2$, $j = 2p$, $k = 2p^2$.

Therefore t_i, t_j and t_k denote respectively the number of generators of order 2, 2p and $2p^2$. Now the product of the odd powers of the elements of order 2, 2p and $2p^2$ does not belong to M' [using (i) of (b) of the list II given in section 3.1]. Therefore we arrived at a contradiction. Hence t_i, t_j and t_k cannot take odd values for the above values of i, j and k.

(ii) : If there are two periods of Γ say i and j, and i, j take values from the set $\{2, 2p, 2p^2\}$ then t_i and t_j must be both even or both odd, otherwise their product will not be in M' [using (a)(i) and c(i) of the list II given in section 3.1]

3.E. (i) : We are to show that if $t_4 \neq 0$ and t_{p^2}, t_{2p^2} are simultaneously zero then for $t_4 = 2$, $\gamma \geq 1$ otherwise $\gamma \geq 0$.

Let $\phi : \Gamma \rightarrow M$ be a smooth epimorphism where

$$\Gamma = (\gamma; 2, ..., 2, 4, ..., 4, p, ..., p, p^2, ..., p^2, 2p, ..., 2p, 2p^2, ..., 2p^2).$$

Let $x_1,...,x_{t_2}, y_1,...,y_{t_4}, z_1,...,z_{t_p}, u_1,...,u_{t_{p^2}}, v_1,...,v_{t_{2p}}, w_1,...w_{t_{2p^2}}$ be the generators of order 2, 4, p, p^2, 2p and $2p^2$ respectively. We can map the finite order generators as follows :

$$\phi(x_i) = b^2;\ i = 1, 2, ..., t_2.$$

$\phi(y_j) = ba^y$, $j=1, 3, \dots, t_4 - 1$ where $0 \leq y < p^2$

$\phi(y_j) = b^3a^y$, $j = 2, 4, \dots, t_4$ where $0 \leq y < p^2$

$\phi(z_k) = a^{pt_k}$, $k = 1, 2, \dots, t_p$.

$\phi(u_\ell) = a^{k_\ell}$, $(k_\ell, p^2) = 1$, $\ell = 1, 2, \dots, t_{p^2}$.

$\phi(v_m) = b^2a^{pt_m}$, $m = 1, 2, \dots, t_{2p}$.

$\phi(w_n) = b^2a^{k_n}$, $(k_n, p^2) = 1$, $n = 1, 2, \dots, t_{2p^2}$.

If possible, let $\gamma = 0$ for $t_4 = 2$ and t_{p^2}, t_{2p^2} both are simultaneously zero.

In this case ϕ will be a homomorphism but not an epimorphism.

In the above mapping if we map the fourth ordered generators of Γ in the following way :

$\phi(y_1) = b$, $\phi(y_2) = ba$

then ϕ will be an epimorphism but not a homomorphism, since it does not satisfy the defining relation (3.2). Hence $\gamma \geq 1$ for $t_4 = 2$ and t_{p^2}, t_{2p^2} both are simultaneously zero.

3 (E) (ii) : The proof is similar to 3.E.(i).

Sufficiency :

Let Γ be a Fuchsian group whose periods and genus satisfy any one of the conditions (1) to (3). Let Γ have a presentation of the form :

$$\Big\langle x_1,\dots,x_{t_2}, y_1,\dots,y_{t_4}, z_1,\dots,z_{t_p}, u_1,\dots,u_{t_{p^2}}, v_1,\dots,v_{t_{2p}}, w_1,\dots,w_{t_{2p^2}}, \alpha_1,\beta_1,\dots,\alpha_\gamma,\beta_\gamma :$$

$$x_1^2=\dots=x_{t_2}^2=y_1^4=\dots=y_{t_4}^4=z_1^p=\dots=z_{t_p}^p=u_1^{p^2}=\dots=u_{t_{p^2}}^{p^2}=v_1^{2p}=v_2^{2p}=\dots=v_{t_{2p}}^{2p}$$

$$=w_1^{2p^2}=\dots=w_{t_{2p^2}}^{2p^2}=\prod_{i=1}^{t_2}x_i\prod_{j=1}^{t_4}y_j\prod_{k=1}^{t_p}z_k\prod_{\ell=1}^{t_{p^2}}u_\ell\prod_{m=1}^{t_{2p}}v_m\prod_{n=1}^{t_{2p^2}}w_n\prod_{i=1}^{\gamma}[\alpha_i,\beta_i]=1\Big\rangle \quad \text{......(3.1.1.2)}$$

where $$2\gamma-2+t_2\left(1-\frac{1}{2}\right)+t_4\left(1-\frac{1}{4}\right)+t_p\left(1-\frac{1}{p}\right)+t_{p^2}\left(1-\frac{1}{p^2}\right)$$

$$+t_{2p}\left(1-\frac{1}{2p}\right)+t_{2p^2}\left(1-\frac{1}{2p^2}\right)>0.$$

That is Γ has the signature :

$(\gamma;\ 2, \dots, 2, 4, \dots, 4, p, \dots, p, p^2, \dots, p^2, 2p, \dots, 2p, 2p^2, \dots, 2p^2)$.

Further t_4 is always even. $t_4 = 0$ implies $\gamma \geq 1$ and if $t_4 \neq 0$ then $\gamma \geq 0$. Also according to the conditions stated in the theorem (3.1.1) γ, t_2, t_p, t_{p^2}, t_{2p}, t_{2p^2} can not take certain values simultaneously.

We now investigate the following cases :

I. For $s = 0$, $\gamma \geq 2$.

Let us define $\phi(\alpha_1) = a = \phi(\beta_1)$

$\phi(\alpha_2) = b = \phi(\beta_2)$

$\phi(\alpha_i) = 1 = \phi(\beta_i)$, $(i = 3, 4, \dots, \gamma)$.

Then ϕ is a homomorphism because the images of the generators of Γ under ϕ satisfy relations identical to the defining relations of Γ given in (3.1.1.2). Also since a and b generate M, ϕ is surjective i.e., ϕ is an epimorphism.

When the periods of Γ satisfy the conditions (2), (3) (B) then,

II. $s = 1$ and (a) $\Gamma = (\gamma;\ p)$, $\gamma \geq 2$

or (b) $\Gamma = (\gamma;\ p^2)$, $\gamma \geq 1$.

We exhibit a smooth epimorphism ϕ from Γ to M in each of the above cases.

For case (a) $\phi(x_1) = a^p$

$\phi(\alpha_1) = b$, $\phi(\beta_1) = a^{py}$, $1 \leq y < p$

$\phi(\alpha_2) = a = \phi(\beta_2)$

$\phi(\alpha_i) = 1 = \phi(\beta_i)$, $i = 3, 4, \ldots, \gamma$.

Now ϕ will be a homomorphism if

$$\phi(x_1)[\phi(\alpha_1),\phi(\beta_1)][\phi(\alpha_2),\phi(\beta_2)]\prod_{i=3}^{\gamma}[\phi(\alpha_i),\phi(\beta_i)]=1$$

$\Rightarrow a^p[b, a^{py}][a, a] \,.\, 1 = 1$

$\Rightarrow a^p(b^{-1}a^{-py}ba^{py}) \,.\, 1 = 1$

$\Rightarrow a^p \,.\, a^{2py} = 1$

$\Rightarrow p(1 + 2y) \equiv 0 \pmod{p^2}$

$\Rightarrow 1 + 2y \equiv 0 \pmod{p}$

$\Rightarrow 2y \equiv -1 \pmod{p}$.

Since $(2, p) = 1$, therefore y has a unique solution.

Hence ϕ is a homomorphism and also a smooth epimorphism, since ϕ preserves the periods of Γ.

For case (b) we define $\phi : \Gamma \to M$ as follows :

$\phi(x_1) = a^{-2}$

$\phi(\alpha_1) = b$, $\phi(\beta_1) = a$

$\phi(\alpha_i) = 1 = \phi(\beta_i)$, $i = 2, 3, \ldots, \gamma$.

It is easy to verify that ϕ satisfy the relation (3.1.1.2) of Γ. So ϕ is a homomorphism which preserves the periods of Γ. Moreover ϕ is onto. Hence ϕ defined above is a smooth epimorphism.

III. When Γ has the signature given in 3(C) of the theorem (3.1.1) a given mapping ϕ from Γ onto M will be a smooth epimorphism if the images of the generators of Γ under ϕ satisfy relations identical to the defining relations of Γ given in (3.1.1.2) and the finite order generators of Γ preserves their orders under ϕ. Let us now exhibit a smooth epimorphism $\phi : \Gamma \to M$ in each of the cases of 3(C) from (i) through (x).

(i) $(\gamma\, ;\ 2,\ 2);\ \gamma \geq 2$

$$\phi(x_1) = b^2,\ \phi(x_2) = b^{-2}$$

$$\phi(\alpha_1) = a = \phi(\beta_1)$$

$$\phi(\alpha_2) = b = \phi(\beta_2)$$

$$\phi(\alpha_i) = 1 = \phi(\beta_i)\ (i = 3, 4, \ldots, \gamma).$$

(ii) $(\gamma\, ;\ 2,\ 2p);\ \gamma \geq 2$

$$\phi(x_1) = b^2,\ \phi(v_1) = b^2a^p$$

$$\phi(\alpha_1) = b,\ \phi(\beta_1) = a^{py},\ 1 \leq y < p$$

$$\phi(\alpha_2) = a = \phi(\beta_2),\ \phi(\alpha_i) = 1 = \phi(\beta_i),\ i = 3, 4, \ldots, \gamma.$$

Now ϕ will be a homomorphism if,

$$\phi(x_1)\phi(v_1)\prod_{i=1}^{\gamma}[\phi(\alpha_i),\phi(\beta_i)] = 1$$

$$\Rightarrow b^2\,.\,b^2a^p[b,\ a^{py}]\ [a,\ a] = 1$$

$$\Rightarrow a^p(b^{-1}a^{-py}b)a^{py}\,.\,1 = 1$$

$$\Rightarrow a^pa^{2py} = 1$$

$$\Rightarrow p(1 + 2y) \equiv 0 \pmod{p^2}$$

$$\Rightarrow 1 + 2y \equiv 0 \pmod{p}$$

$$\Rightarrow 2y \equiv -1 \pmod{p}.$$

The above congruence has a unique solution. It is obvious that ϕ is a smooth epimorphism.

(iii) $(\gamma\, ;\ 2,\ 2p^2);\ \gamma \geq 1$

$$\phi(x_1) = b^2,\ \phi(w_1) = b^2a^2$$

$$\phi(\alpha_1) = ab,\ \phi(\beta_1) = a^{-1}$$

$$\phi(\alpha_i) = 1 = \phi(\beta_i),\ i = 2, 3, \ldots, \gamma.$$

It is obvious that ϕ is a smooth epimorphism.

(iv) $(\gamma ; 4, 4);\ \gamma \geq 1$

$$\phi(y_1) = b,\ \phi(y_2) = b^{-1}$$

$$\phi(\alpha_1) = a = \phi(\beta_1),\ \phi(\alpha_i) = 1 = \phi(\beta_i);\ i = 2, 3, \ldots, \gamma.$$

It is easy to verify that ϕ defined as above is a smooth epimorphism.

(v) $(\gamma ; p, p);\ \gamma \geq 2$

$$\phi(z_1) = a^p,\ \phi(z_2) = a^{-p}$$

$$\phi(\alpha_1) = a = \phi(\beta_1)$$

$$\phi(\alpha_2) = b = \phi(\beta_2)$$

$$\phi(\alpha_i) = 1 = \phi(\beta_i);\ i = 3, 4, \ldots, \gamma.$$

(vi) $(\gamma ; p, p^2);\ \gamma \geq 1$

$$\phi(z_1) = a^p$$

$$\phi(u_1) = a^{-2}$$

$$\phi(\alpha_1) = b,\ \phi(\beta_1) = a^{py+1},\ 1 \leq y < p$$

$$\phi(\alpha_i) = 1 = \phi(\beta_i);\ i = 2, 3, \ldots, \gamma.$$

Now ϕ will be a homomorphism if

$$\phi(z_1)\phi(u_1)[\phi(\alpha_1),\phi(\beta_1)]\prod_{i=2}^{\gamma}[\phi(\alpha_i),\phi(\beta_i)] = 1$$

$$\Rightarrow a^p a^{-2}[b, a^{py+1}] \,.\, 1 = 1$$

$$\Rightarrow a^{p-2} b^{-1} a^{-(py+1)} b a^{py+1} = 1$$

$$\Rightarrow a^{p-2} a^{2(py+1)} = 1$$

$$\Rightarrow p + 2py \equiv 0 \pmod{p^2}$$

$$\Rightarrow (1 + 2y) \equiv 0 \pmod{p}$$

$$\Rightarrow 2y \equiv -1 \pmod{p}.$$

Since $(2, p) = 1$, therefore y has a unique solution. Hence ϕ is a homomorphism and also a smooth epimorphism since ϕ preserves the periods of Γ.

(vii) $(\gamma ; p^2, p^2);\ \gamma \geq 1$

$$\phi(u_1) = a,\ \phi(u_2) = a^{-1}$$

$$\phi(\alpha_1) = b = \phi(\alpha_2)$$

$$\phi(\alpha_i) = 1 = \phi(\beta_i);\ i = 2, 3, \dots, \gamma.$$

It is obvious that ϕ is a smooth epimorphism.

(viii) $(\gamma\,;\ 2p,\ 2p);\ \gamma \geq 2$

$$\phi(v_1) = b^2a^{pt},\ 1 \leq t < p$$

$$\phi(v_2) = (b^2a^{pt})^{-1},\ 1 \leq t < p$$

$$\phi(\alpha_1) = a = \phi(\beta_1),\ \phi(\alpha_2) = b = \phi(\beta_2)$$

$$\phi(\alpha_i) = 1 = \phi(\beta_i);\ i = 3, 4, \dots, \gamma.$$

Obviously ϕ is a smooth epimorphism.

(ix) $(\gamma\,;\ 2p,\ 2p^2);\ \gamma \geq 1$

$$\phi(v_1) = b^2a^{2py_1}$$

$$\phi(w_1) = b^2a^y,\ (y, p) = 1$$

$$\phi(\alpha_1) = b,\ \phi(\beta_1) = a^{-(py_1+1)}, 1 \leq y < p.$$

Now ϕ will be a homomorphism if,

$$\phi(v_1)\phi(w_1)[\phi(\alpha_1),\ \phi(\beta_1)] = 1$$

$$\Rightarrow b^2a^{2py_1}b^2a^y[b, a^{-(py_1+1)}] = 1$$

$$\Rightarrow a^{2py_1+y}a^{-2(py_1+1)} = 1$$

$$\Rightarrow a^{y-2} = 1$$

$$\Rightarrow y \equiv 2 \text{ (mop } p^2).$$

The above congruence has a unique solution. Hence ϕ is a homomorphism and also a smooth epimorphism since ϕ preserves the periods of Γ.

(x) $(\gamma\,;\ 2p^2,\ 2p^2);\ \gamma \geq 1$

$$\phi(w_1) = b^2a^k;\ (k, p) = 1$$

$$\phi(w_2) = (b^2a^k)^{-1};\ (k, p) = 1$$

$$\phi(\alpha_1) = b = \phi(\beta_1),\ \phi(\alpha_i) = 1 = \phi(\beta_i);\ i = 2, 3, \dots, \gamma.$$

It is easy to verify that ϕ defined as above is a smooth epimorphism.

(IV) When the periods of Γ satisfy the conditions 3(A), 3(D)(a),(b),(c) of the theorem

(3.1.1) then for $s \geq 3$ we shall investigate the following possible cases :

1. s is odd and $t_4 = 0$, $\gamma \geq 1$, then

(a) t_p odd and t_2, t_{p^2}, t_{2p}, t_{2p^2} all are (non-zero) even.

(a_1) If t_{p^2}, t_{2p^2} both are simultaneously zero then $\gamma \geq 2$.

(b) t_{p^2} odd and t_2, t_p, t_{2p}, t_{2p^2} all are even.

(c) t_2, t_p, t_{2p} all are odd and t_{p^2}, t_{2p^2} both are (non-zero) even.

(c_1) if t_{p^2}, t_{2p^2} both are simultaneously zero then $\gamma \geq 2$.

(d) t_2, t_p, t_{2p^2} all are odd and t_{p^2}, t_{2p} are even.

(e) t_2, t_{p^2}, t_{2p} all are odd and t_p, t_{2p^2} both are even.

(f) t_2, t_{p^2}, t_{2p^2} all are odd and t_p, t_{2p} both are even.

(g) t_p, t_{2p}, t_{2p^2} all are odd and t_2, t_{p^2} both are even.

(h) t_{p^2}, t_{2p}, t_{2p^2} all are odd and t_2, t_p both are even.

The following cases are untenable:

(i) t_2 odd and t_p, t_{p^2}, t_{2p}, t_{2p^2} all are even.

(ii) t_2, t_p, t_{p^2} all are odd and t_{2p}, t_{2p^2} both are even.

(iii) t_p, t_{p^2}, t_{2p} all are odd and t_2, t_{2p^2} both are even.

(iv) t_p, t_{p^2}, t_{2p^2} all are odd and t_2, t_{2p} both are even.

2. s is even, $t_4 = 0$, $\gamma \geq 1$ then

(a) t_2, t_p, t_{p^2}, t_{2p}, t_{2p^2} all are (non-zero) even.

(a_1) If t_{p^2}, t_{2p^2} both simultaneously zero then $\gamma \geq 2$.

(b) t_2, t_{2p} both are odd and t_p, t_{p^2}, t_{2p^2} all are (non-zero) even.

(b_1) If t_{p^2}, t_{2p^2} both are simultaneously zero then $\gamma \geq 2$.

(c) t_2, t_{2p^2} both are odd and t_p, t_{p^2}, t_{2p} all are even.

(d) t_p, t_{p^2} both are odd and t_2, t_{2p}, t_{2p^2} all are even.

(e) t_{2p}, t_{2p^2} both are odd and t_2, t_p, t_{p^2} all are even.

(f) t_2, t_p, t_{p^2}, t_{2p} all are odd and t_{2p^2} even.

(g) t_2, t_p, t_{p^2}, t_{2p^2} all are odd and t_{2p} even.

(h) t_p, t_{p^2}, t_{2p}, t_{2p^2} all are odd and t_2 even.

The following cases are untenable :

(i) t_2, t_p both are odd and t_{p^2}, t_{2p}, t_{2p^2} all are even.

(ii) t_2, t_{p^2} both are odd and t_p, t_{2p}, t_{2p^2} all are even.

(iii) t_p, t_{2p} both are odd and t_2, t_{p^2}, t_{2p^2} all are even.

(iv) t_p, t_{2p^2} both are odd and t_2, t_{p^2}, t_{2p} all are even.

(v) t_{p^2}, t_{2p} both are odd and t_2, t_p, t_{2p^2} all are even.

(vi) t_{p^2}, t_{2p^2} both are odd and t_2, t_p, t_{2p} all are even.

(vii) t_2, t_{p^2}, t_{2p}, t_{2p^2} all are odd and t_p even.

When the periods of Γ satisfy the conditions 3(E) (i) and (ii) of the theorem (3.1.1) then the above conditions will also be sufficient for construction of smooth epimorphism from Γ onto M with $\gamma \geq 0$ except for the case when t_{p^2} and t_{2p^2} both are simultaneously zero. In this case $\gamma \geq 1$.

In each of the cases listed above a given mapping ϕ from Γ onto M will be a **smooth epimorphism** if the images of the generators of Γ under ϕ satisfy relations identical to the defining relations of Γ given in (3.1.1.2) and the finite order generators of Γ preserves their orders under ϕ. Let us now exhibit a smooth epimorphism $\phi : \Gamma \rightarrow M$ in each of the cases listed above.

(1)(a) $t_4 = 0$; t_p odd and $t_2, t_{p^2}, t_{2p}, t_{2p^2}$ all are non-zero even and $\gamma \geq 1$.

$\phi(x_i) = b^2$, $i = 1, 3, \ldots, t_2 - 1$.

$\phi(x_i) = b^{-2}$, $i = 2, 4, \ldots, t_2$.

$\phi(z_1) = a^{2p}$, $\phi(z_i) = a^{pt}$, $1 \leq t < p$ and $i = 2, 4, \ldots, t_p - 1$.

$\phi(z_i) = a^{-pt}$, $1 \leq t < p$ and $i = 3, 5, \ldots, t_p$.

$\phi(u_i) = a$, $i = 1, 3, \ldots, t_{p^2} - 1$.

$\phi(u_i) = a^{-1}, i = 2, 4, \ldots, t_{p^2}$.

$\phi(v_i) = b^2a^{pt}$, $1 \leq t < p$ and $i = 1, 3, \ldots, t_{2p} - 1$.

$\phi(v_i) = (b^2a^{pt})^{-1}$, $1 \leq t < p$ and $i = 2, 4, \ldots, t_{2p}$.

$\phi(w_i) = b^2a^k$, $(k, p) = 1$ and $i = 1, 3, \ldots, t_{2p^2} - 1$.

$\phi(w_i) = (b^2a^k)^{-1}$, $(k, p) = 1$ and $i = 2, 4, \ldots, t_{2p^2}$.

$\phi(\alpha_1) = b$, $\phi(\beta_1) = a^{-p}$.

$\phi(\alpha_i) = 1 = \phi(\beta_i)$; $i = 2, 3, \ldots, \gamma$.

Now, $$\prod_{i=1}^{t_2}\phi(x_i)\phi(z_1)\prod_{i=2}^{t_p}\phi(z_i)\prod_{i=1}^{t_{p^2}}\phi(u_i)\prod_{i=1}^{t_{2p}}\phi(v_i)\prod_{i=1}^{t_{2p^2}}\phi(w_i)\prod_{i=1}^{\gamma}[\phi(\alpha_i),\phi(\beta_i)]$$

$$= b^2b^{-2} \ldots b^2b^{-2}a^{2p}a^{pt}a^{-pt} \ldots a^{pt}a^{-pt}\, aa^{-1} \ldots aa^{-1}(b^2a^{pt})(b^2a^{pt})^{-1}$$

$$\ldots (b^2a^{pt})(b^2a^{pt})^{-1}(b^2a^k)(b^2a^k)^{-1} \ldots (b^2a^k)(b^2a^k)^{-1}(b^{-1}a^pba^{-p}),$$

with ${}^{t_2}\!/_2$ occurrences of b^2b^{-2}, ${}^{(t_p-1)}\!/_2$ occurrences of $a^{pt}a^{-pt}$, ${}^{t_{p^2}}\!/_2$ occurrences of aa^{-1}, ${}^{t_{2p}}\!/_2$ occurrences of $(b^2a^{pt})(b^2a^{pt})^{-1}$, ${}^{t_{2p^2}}\!/_2$ occurrences of $(b^2a^k)(b^2a^k)^{-1}$, so that

$$\prod_{i=1}^{t_2}\phi(x_i)\phi(z_1)\prod_{i=2}^{t_p}\phi(z_i)\prod_{i=1}^{t_{p^2}}\phi(u_i)\prod_{i=1}^{t_{2p}}\phi(v_i)\prod_{i=1}^{t_{2p^2}}\phi(w_i)[b,a^{-p}]=1.$$

Therefore ϕ is a homomorphism, also since a and b generate M, ϕ is onto. ϕ preserves the orders of the finite order generators of Γ, hence ϕ is a smooth epimorphism.

Note : If in the above case $t_4 \neq 0$ then $\gamma \geq 0$.

In the above mapping in place of the mapping of infinite order generators we map the fourth order generators of Γ as follows :

$\phi(y_1) = (ba)^{-1}$, $\phi(y_2) = ba^k$, $(k, p) = 1$,

$\phi(y_i) = b^{-1}$; $i = 3, 5, \ldots, t_4 - 1$,

$\phi(y_i) = b$; $i = 4, 6, \ldots, t_4$,

and we get $k \equiv -(2p - 1) \pmod{p^2}$

which has a solution. Therefore ϕ is a smooth epimorphism since ϕ preserves the periods of Γ.

(a_1) If t_{p^2}, t_{2p^2} both are simultaneously zero then $\gamma \geq 2$.

In the above homomorphism [1(a)] we can map

$\phi(\alpha_2) = a = \phi(\beta_2)$

and $\phi(\alpha_i) = 1 = \phi(\beta_i)$ for $i = 3, 4, \ldots, \gamma$

so that ϕ will be an onto homomorphism.

Note : In the above case when,

(1) $t_4 = 2$ and t_{p^2}, t_{2p^2} are simultaneously zero then $\gamma \geq 1$.

We can construct the above homomorphism [1(a)] from Γ onto M by replacing $\phi(u_i)$ and $\phi(w_i)$ by :

$$\phi(y_1) = (ba)^{-1} \text{ and } \phi(y_2) = ba$$

and the other elements are mapped as above so that ϕ will be an onto homomorphism.

(2) $t_4 \geq 4$ and t_{p^2}, t_{2p^2} are simultaneously zero then $\gamma \geq 0$.

The above homomorphism [1(a)] can be constructed from Γ onto M as follows :

$\phi(y_1) = b$

$\phi(y_2) = b^3a^k,\ 0 \leq k < p^2$

$\phi(y_3) = (ba)^{-1},\ \phi(y_4) = b$

$\phi(y_i) = b;\ i = 5, 7, \ldots, t_4 - 1$

$\phi(y_i) = b^{-1};\ i = 6, 8, \ldots, t_4,$

so that ϕ will be an onto homomorphism and getting,

$$k \equiv -(2p - 1) \pmod{p^2}$$

which has a solution.

(3) $t_4 \geq 4$ and one of t_{p^2}, t_{2p^2} is zero then $\gamma \geq 0$.

The fourth ordered generators can be mapped as follows :

$\phi(y_1) = (ba)^{-1},\ \phi(y_2) = ba^k,\ (k, p) = 1$

$\phi(y_i) = b,\ i = 3, 5, \ldots, t_4 - 1$

$\phi(y_i) = b^{-1},\ i = 4, 6, \ldots, t_4$

so that ϕ will be a homomorphism and thus getting,

$$k \equiv -(2p - 1) \pmod{p^2}$$

which has a solution.

(b) $t_4 = 0$; t_{p^2} odd and t_2, t_p, t_{2p}, t_{2p^2} all are even then $\gamma \geq 1$.

$\phi(x_i) = b^2$, $i = 1, 3, ..., t_2 - 1$.

$\phi(x_i) = b^{-2}$, $i = 2, 4, ..., t_2$.

$\phi(z_i) = a^p$, $i = 1, 3, ..., t_p - 1$.

$\phi(z_i) = a^{-p}$, $i = 2, 4, ..., t_p$.

$\phi(u_1) = a^2$, $\phi(u_i) = a$, $i = 2, 4, ..., t_{p^2} - 1$.

$\phi(u_i) = a^{-1}$, $i = 3, 5, ..., t_{p^2}$.

$\phi(v_i) = b^2a^{pt}$, $i = 1, 3, ..., t_{2p} - 1$.

$\phi(v_i) = (b^2a^{pt})^{-1}$, $i = 2, 4, ..., t_{2p}$.

$\phi(w_i) = b^2a^k$, $(k, p) = 1$ and $i = 1, 3, ..., t_{2p^2} - 1$.

$\phi(w_i) = (b^2a^k)^{-1}$, $(k, p) = 1$ and $i = 2, 4, ..., t_{2p^2}$.

$\phi(\alpha_1) = a$, $\phi(\beta_1) = b$;

$\phi(\alpha_i) = 1 = \phi(\beta_i)$; $i = 2, 3, ..., \gamma$.

Now ϕ will be a homomorphism if,

$$\prod_{i=1}^{t_2}\phi(x_i)\prod_{i=1}^{t_p}\phi(z_i)\phi(u_1)\prod_{i=2}^{t_{p^2}}\phi(u_i)\prod_{i=1}^{t_{2p}}\phi(v_i)\prod_{i=1}^{t_{2p^2}}\phi(w_i)\prod_{i=1}^{\gamma}[\phi(\alpha_i),\phi(\beta_i)] = 1.$$

Now L.H.S.

$$= b^2b^{-2} \ldots b^2b^{-2} \,.\, a^pa^{-p} \ldots a^pa^{-p} \,.\, a^2aa^{-1} \ldots aa^{-1} \,.\, (b^2a^{pt})(b^2a^{pt})^{-1}$$

$$\ldots (b^2a^{pt})(b^2a^{pt})^{-1} \,.\, (b^2a^k)(b^2a^k)^{-1} \ldots (b^2a^k)(b^2a^k)^{-1} \,.\, a^{-1}b^{-1}ab$$

with $^{t_2}/_2$ occurrences of the product b^2b^{-2}, $^{t_p}/_2$ occurrences of the product a^pa^{-p}, $^{(t_{p^2}-1)}/_2$ occurrences of the product aa^{-1}, $^{t_{2p}}/_2$ occurrences of the product $(b^2a^{pt})(b^2a^{pt})^{-1}$, $^{t_{2p^2}}/_2$ occurrences of the product $(b^2a^k)(b^2a^k)^{-1}$ and with $a^{-1}b^{-1}ab = a^{-2}$ so that

$$\prod_{i=1}^{t_2}\phi(x_i)\prod_{i=1}^{t_p}\phi(z_i)\phi(u_1)\prod_{i=2}^{t_{p^2}}\phi(u_i)\prod_{i=1}^{t_{2p}}\phi(v_i)\prod_{i=1}^{t_{2p^2}}\phi(w_i)\prod_{i=1}^{\gamma}[\phi(\alpha_i),\phi(\beta_i)]=1.$$

Therefore ϕ is a homomorphism, also since a and b generate M, ϕ is onto. ϕ preserves the order of the finite order generators of Γ, hence ϕ is a smooth epimorphism.

Note : In the above case if $t_4 \neq 0$ then $\gamma \geq 0$.

In the above mapping in place of the infinite order generators we map the fourth order generators of Γ as follows :

$\phi(y_1) = b$, $\phi(y_2) = b^3a^k$, $0 \leq k < p^2$

$\phi(y_i) = b^{-1}$; $i = 3, 5, \ldots, t_4 - 1$,

$\phi(y_i) = b$; $i = 4, 6, \ldots, t_4$,

and then we get $\quad k + 2 \equiv 0 \pmod{p^2}$

i.e., $\quad k \equiv -2 \pmod{p^2}$

which has a solution. Therefore ϕ is a smooth epimorphism since ϕ preserves the periods of Γ.

(c) $t_4 = 0$; t_2, t_p, t_{2p} all are odd and t_{p^2}, t_{2p^2} both are (non-zero) even then $\gamma \geq 1$.

$\phi(x_1) = b^2$, $\phi(x_i) = b^{-2}$, $i = 2, 4, 6, \ldots, t_2 - 1$.

$\phi(x_i) = b^2$, $i = 3, 5, 7, \ldots, t_2$.

$\phi(z_1) = a^p$, $\phi(z_i) = a^p$, $i = 2, 4, \ldots, t_p - 1$.

$\phi(z_i) = a^{-p}$, $i = 3, 5, \ldots, t_p$.

$\phi(u_i) = a$, $i = 1, 3, \ldots, t_{p^2} - 1$.

$\phi(u_i) = a^{-1}$, $i = 2, 4, \ldots, t_{p^2}$.

$\phi(v_1) = b^2a^p$, $\phi(v_i) = b^2a^{pt}$, $1 \leq t < p$ and $i = 2, 4, \ldots, t_{2p}-1$.

$\phi(v_i) = (b^2a^{pt})^{-1}$, $i = 3, 5, \ldots, t_{2p}$.

$\phi(w_i) = b^2a^k$, $(k, p) = 1$ and $i = 1, 3, \ldots, t_{2p^2} - 1$.

$\phi(w_i) = (b^2a^k)^{-1}$, $(k, p) = 1$ and $i = 2, 4, \ldots, t_{2p^2}$.

$\phi(\alpha_1) = b$, $\phi(\beta_1) = a^{-p}$, $(k, p) = 1$.

$\phi(\alpha_i) = 1 = \phi(\beta_i)$, $i = 2, 3, \ldots, \gamma$.

Now $$\phi(x_1)\prod_{i=2}^{t_2}\phi(x_i)\phi(z_1)\prod_{i=2}^{t_p}\phi(z_i)\prod_{i=1}^{t_{p^2}}\phi(u_i)\phi(v_1)\prod_{i=2}^{t_{2p}}\phi(v_i)\prod_{i=1}^{t_{2p^2}}\phi(w_i)\prod_{i=1}^{\gamma}[\phi(\alpha_i),\phi(\beta_i)]$$

$$= b^2(b^{-2}b^2 \ldots b^{-2}b^2)(a^p)(a^pa^{-p} \ldots a^pa^{-p})(aa^{-1} \ldots aa^{-1})b^2a^pb^2a^{pt}(b^2a^{pt})^{-1}$$
$$\ldots (b^2a^{pt})(b^2a^{pt})^{-1}b^2a^k(b^2a^k)^{-1} \ldots (b^2a^k)(b^2a^k)^{-1}[b, a^{-p}] \,.\, 1$$

with $t_2 - 1/2$ occurrences of $b^{-2}b^2$, $(t_p - 1)/2$ occurrences of $a^p.a^{-p}$, $t_{p^2}/2$ occurrences of aa^{-1}, $t_{2p} - 1/2$ occurrences of $(b^2a^{pt})(b^2a^{pt})^{-1}$, $t_{2p^2}/2$ occurrences of $(b^2a^k)(b^2a^k)^{-1}$ so that

$$\prod_{i=1}^{t_2}\phi(x_i)\prod_{i=1}^{t_p}\phi(z_i)\prod_{i=1}^{t_{p^2}}\phi(u_i)\prod_{i=1}^{t_{2p}}\phi(v_i)\prod_{i=1}^{t_{2p^2}}\phi(w_i)\prod_{i=1}^{\gamma}[\phi(\alpha_i),\phi(\beta_i)] = 1.$$

Therefore ϕ is a homomorphism and also a smooth epimorphism, since ϕ preserves the periods of Γ.

Note : If in the above case $t_4 \neq 0$ then $\gamma \geq 0$.

In the above mapping in place of the infinite order generators we map the fourth order generators of Γ as follows :

$\phi(y_1) = (ba)^{-1}$, $\phi(y_2) = ba^k$, $(k, p) = 1$

$\phi(y_i) = b$; $i = 3, 5, \ldots, t_4 - 1$,

$\phi(y_i) = b^{-1}$; $i = 4, 6, \ldots, t_4$,

so that ϕ will be a homomorphism and also ϕ preserves the order of the finite order generators of Γ, hence ϕ is a smooth epimorphism.

(c_1) $t_4 = 0$ and if t_{p^2}, t_{2p^2} both are simultaneously zero then $\gamma \geq 2$.

In the above homomorphism [1(c)] we can map $\phi(\alpha_2) = a = \phi(\beta_2)$ and $\phi(\alpha_i) = 1 = \phi(\beta_i)$ for $i = 3, 4, \ldots, \gamma$ so that ϕ will be an onto homomorphism.

Note : In the above case when

***(1)* $t_4 = 2$ *and* t_{p^2}, t_{2p^2} *are simultaneously zero then* $\gamma \geq 1$.**

We can construct the above homomorphism [1(c)] from Γ onto M by replacing $\phi(u_i)$ and $\phi(w_i)$ by $\phi(y_1) = (ba)^{-1}$, $\phi(y_2) = b$ and infinite order generators can be mapped as $\phi(\alpha_1) = b$, $\phi(\beta_1) = a^k$ so that ϕ will be a smooth epimorphism.

***(2)* $t_4 \geq 4$ *and* t_{p^2}, t_{2p^2} *simultaneously zero then* $\gamma \geq 0$.**

The above homomorphism [1(c)] can be constructed from Γ onto M as follows :

$\phi(y_1) = b$, $\phi(y_2) = b^3a^k$, $0 \leq k < p^2$

$\phi(y_3) = (ba)^{-1}$, $\phi(y_4) = b$

$\phi(y_i) = b^{-1}$, $i = 5, 7, \ldots, t_4 - 1$

$\phi(y_i) = b$, $i = 6, 8, \ldots, t_4$.

So that ϕ will be a homomorphism and also a smooth epimorphism.

***(3)* $t_4 \geq 4$ *and one of* t_{p^2}, t_{2p^2} *is zero then* $\gamma \geq 0$.**

The fourth ordered generators can be mapped as follows :

$\phi(y_1) = ba^{-1}$, $\phi(y_2) = b^3a^k$, $0 \leq k < p^2$.

$\phi(y_i) = b$, $i = 3, 5, \ldots, t_4 - 1$.

$\phi(y_i) = b^{-1}$, $i = 4, 6, \ldots, t_4$,

so that ϕ will be a homomorphism and thus getting

$k \equiv -(2p + 1)(\text{mod } p^2)$, which has a solution.

Also ϕ is a smooth epimorphism since ϕ preserves the periods of Γ.

***(d)* $t_4 = 0$; t_2, t_p, t_{2p^2} *all are odd and* t_{p^2}, t_{2p} *are even then* $\gamma \geq 1$.**

$\phi(x_1) = b^2$, $\phi(x_i) = b^{-2}$, $i = 2, 4, 6, \ldots, t_2 - 1$.

$\phi(x_i) = b^2$, $i = 3, 5, 7, \ldots, t_2$.

$\phi(z_1) = a^p$, $\phi(z_i) = a^{-p}$, $i = 2, 4, 6, \ldots, t_p - 1$.

$\phi(z_i) = a^p$, $i = 3, 5, 7, \ldots, t_p$.

$\phi(u_i) = a, \ i = 1, 3, 5, \ldots, t_{p^2} - 1.$

$\phi(u_i) = a^{-1}, \ i = 2, 4, 6, \ldots, t_{p^2}.$

$\phi(v_i) = b^2a^{pt}, \ 1 \leq t < p$ and $i = 1, 3, 5, \ldots, t_{2p} - 1.$

$\phi(v_i) = (b^2a^{pt})^{-1}, \ 1 \leq t < p$ and $i = 2, 4, 6, \ldots, t_{2p}.$

$\phi(w_1) = b^2a.$

$\phi(w_i) = b^2a^k, \ (k, p) = 1$ and $i = 2, 4, \ldots, t_{2p^2} - 1.$

$\phi(w_i) = (b^2a^k)^{-1}, \ (k, p) = 1$ and $i = 3, 5, \ldots, t_{2p^2}.$

$\phi(\alpha_1) = b, \ \phi(\beta_1) = a^k, \ (k, p) = 1.$

$\phi(\alpha_i) = 1 = \phi(\beta_i), \ i = 2, 3, \ldots, \gamma.$

Now $$\phi(x_1)\prod_{i=2}^{t_2}\phi(x_i)\phi(z_1)\prod_{i=2}^{t_p}\phi(z_i)\prod_{i=1}^{t_{p^2}}\phi(u_i)\prod_{i=1}^{t_{2p}}\phi(v_i)\phi(w_1)\prod_{i=2}^{t_{2p^2}}\phi(w_i)\prod_{i=1}^{\gamma}[\phi(\alpha_i),\phi(\beta_i)]$$

$$= b^2(b^{-2}b^2 \ldots b^{-2}b^2)a^p(a^{-p}a^p \ldots a^{-p}a^p)(aa^{-1} \ldots aa^{-1})(b^2a^{pt}(b^2a^{pt})^{-1} \ldots$$
$$\ldots (b^2a^{pt})(b^2a^{pt})^{-1}b^2ab^2a^k(b^2a^K)^{-1} \ldots (b^2a^k)(b^2a^k)^{-1})[b, a^k]$$

with $(t_2 - 1)/2$ occurrences of the product $b^{-2}b^2$, $(t_p - 1)/2$ occurrences of the product $a^{-p}a^p$, $t_{p^2}/2$ occurrences of aa^{-1}, $t_{2p}/2$ occurrences of the product $(b^2a^{pt})(b^2a^{pt})^{-1}$, $(t_{2p^2} - 1)/2$ occurrences of the product $(b^2a^k)(b^2a^k)^{-1}$ so that

$$\prod_{i=1}^{t_2}\phi(x_i)\prod_{i=1}^{t_p}\phi(z_i)\prod_{i=1}^{t_{p^2}}\phi(u_i)\prod_{i=1}^{t_{2p}}\phi(v_i)\prod_{i=1}^{t_{2p^2}}\phi(w_i)\prod_{i=1}^{\gamma}[\phi(\alpha_i),\phi(\beta_i)] = 1$$

gives $2k \equiv -(p + 1) \pmod{p^2}$, which has a unique solution as $(2, p^2) = 1$.

Therefore ϕ is a homomorphism and also smooth epimorphism, since ϕ preserves the order of the finite order generators of Γ.

Note : In the above case if $t_4 \neq 0$ then $\gamma \geq 0$.

In the above mapping in place of the infinite order generators we map the fourth order generators of Γ as follows :

$\phi(y_1) = b$, $\phi(y_2) = b^3a^k$, $0 \leq k < p^2$,

$\phi(y_i) = b$, $i = 3, 5, \ldots, t_4 - 1$,

$\phi(y_i) = b^{-1}$, $i = 4, 6, \ldots, t_4$,

and then we get $k \equiv -(p + 1) \pmod{p^2}$ which has a solution. Therefore ϕ is a smooth epimorphism since ϕ preserves the periods of Γ.

(e) $t_4 = 0$; t_2, t_{p^2}, t_{2p} all are odd and t_p, t_{2p^2} both are even then $\gamma \geq 1$.

$\phi(x_1) = b^2$, $\phi(x_i) = b^2$, $i = 2, 4, 6, \ldots, t_2 - 1$.

$\phi(x_i) = b^{-2}$, $i = 3, 5, \ldots, t_2$.

$\phi(z_i) = a^p$, $i = 1, 3, \ldots, t_p - 1$.

$\phi(z_i) = a^{-p}$, $i = 2, 4, \ldots, t_p$.

$\phi(u_1) = a$, $\phi(u_i) = a^{-1}$, $i = 2, 4, \ldots, t_{p^2} - 1$.

$\phi(u_i) = a$, $i = 3, 5, \ldots, t_{p^2}$.

$\phi(v_1) = b^2a^p$, $\phi(v_i) = b^2a^{pt}$, $1 \leq t < p$ and $i = 2, 4, \ldots, t_{2p} - 1$.

$\phi(v_i) = (b^2a^{pt})^{-1}$, $1 \leq t < p$ and $i = 3, 5, \ldots, t_{2p}$.

$\phi(w_i) = b^2a^k$, $(k, p) = 1$ and $i = 1, 3, \ldots, t_{2p^2} - 1$.

$\phi(w_i) = (b^2a^k)^{-1}$, $(k, p) = 1$ and $i = 2, 4, \ldots, t_{2p^2}$.

$\phi(\alpha_1) = b$, $\phi(\beta_1) = a^k$, $(k, p) = 1$.

$\phi(\alpha_i) = 1 = \phi(\beta_i)$; $i = 2, 3, \ldots, \gamma$.

Now $$\phi(x_1)\prod_{i=2}^{t_2}\phi(x_i)\prod_{i=1}^{t_p}\phi(z_i)\phi(u_1)\prod_{i=2}^{t_{p^2}}\phi(u_i)\phi(v_1)\prod_{i=2}^{t_{2p}}\phi(v_i)\prod_{i=1}^{t_{2p^2}}\phi(w_i)\prod_{i=1}^{\gamma}[\phi(\alpha_i),\phi(\beta_i)]$$

$$= b^2(b^2b^{-2} \ldots b^2b^{-2})(a^pa^{-p} \ldots a^pa^{-p})a(a^{-1}a \ldots a^{-1}a)b^2a^p((b^2a^{pt})(b^2a^{pt})^{-1} \ldots$$
$$\ldots (b^2a^{pt})(b^2a^{pt})^{-1}(b^2a^K)(b^2a^k)^{-1} \ldots (b^2a^k)(b^2a^k)^{-1})[b, a^k]$$

with ${}^{(t_2 - 1)}\!/_2$ occurrences of the product b^2b^{-2}, ${}^{t_p}\!/_2$ occurrences of the product a^pa^{-p}, ${}^{(t_{p^2} - 1)}\!/_2$ occurences of the product $a^{-1}a$, ${}^{(t_{2p} - 1)}\!/_2$ occurrences of the product $(b^2a^{pt})(b^2a^{pt})^{-1}$, ${}^{t_{2p^2}}\!/_2$ occurrences of the product $(b^2a^k)(b^2a^k)^{-1}$ so that

$$\prod_{i=1}^{t_2}\phi(x_i)\prod_{i=1}^{t_p}\phi(z_i)\prod_{i=1}^{t_{p^2}}\phi(u_i)\prod_{i=1}^{t_{2p}}\phi(v_i)\prod_{i=1}^{t_{2p^2}}\phi(w_i)\prod_{i=1}^{\gamma}[\phi(\alpha_i),\phi(\beta_i)] = 1$$

gives $2k \equiv -(p + 1) \pmod{p^2}$, which has a unique solution as $(2, p^2) = 1$.

Therefore ϕ is a homomorphism, also since a and b generate M, ϕ is onto. ϕ preserves the order of the finite order generators of Γ, hence ϕ is a smooth epimorphism.

Note : In the above case if $t_4 \neq 0$ then $\gamma \geq 0$.

In the above mapping in place of the infinite order generators we map the fourth order generators of Γ as follows :

$\phi(y_1) = b$, $\phi(y_2) = b^3a^k$, $0 \leq k < p^2$,

$\phi(y_i) = b$, $i = 3, 5, \ldots, t_4 - 1$,

$\phi(y_i) = b^{-1}$, $i = 4, 6, \ldots, t_4$,

and then we get $k \equiv -(p + 1) \pmod{p^2}$ which has a solution. Therefore ϕ is a smooth epimorphism since ϕ preserves the periods of Γ.

(f) $t_4 = 0$; t_2, t_{p^2}, t_{2p^2} all are odd and t_p, t_{2p} both are even then $\gamma \geq 1$

$\phi(x_1) = b^2$, $\phi(x_i) = b^2$, $i = 2, 4, 6, \ldots, t_2 - 1$.

$\phi(x_i) = b^{-2}$, $i = 3, 5, \ldots, t_2$.

$\phi(z_i) = a^p$, $i = 1, 3, \ldots, t_p - 1$.

$\phi(z_i) = a^{-p}$, $i = 2, 4, \ldots, t_p$.

$\phi(u_1) = a^{-2}$, $\phi(u_i) = a^{-1}$, $i = 2, 4, \ldots, t_{p^2} - 1$.

$\phi(u_i) = a$, $i = 3, 5, \ldots, t_{p^2}$.

$\phi(v_i) = b^2a^{pt}$, $i = 1, 3, \ldots, t_{2p} - 1$.

$\phi(v_i) = (b^2a^{pt})^{-1}$, $i = 2, 4, \ldots, t_{2p}$.

$\phi(w_1) = b^2a$, $\phi(w_i) = b^2a^k$, $(k, p) = 1$ and $i = 2, 4, \ldots, t_{2p^2} - 1$.

$\phi(w_i) = (b^2a^k)^{-1}$, $(k, p) = 1$ and $i = 3, \ldots, t_{2p^2}$.

$\phi(\alpha_1) = b$, $\phi(\beta_1) = a^k$, $(k, p) = 1$.

$\phi(\alpha_i) = 1 = \phi(\beta_i)$, $i = 2, 3, \ldots, \gamma$.

Now, $$\phi(x_1)\prod_{i=2}^{t_2}\phi(x_i)\prod_{i=1}^{t_p}\phi(z_i)\phi(u_1)\prod_{i=2}^{t_{p^2}}\phi(u_i)\prod_{i=1}^{t_{2p}}\phi(v_i)\phi(w_1)\prod_{i=2}^{t_{2p^2}}\phi(w_i)\prod_{i=1}^{\gamma}[\phi(\alpha_i),\phi(\beta_i)]$$

$$= b^2(b^2b^{-2} \ldots b^2b^{-2})(a^pa^{-p} \ldots a^pa^{-p})a^{-2}(a^{-1}a \ldots a^{-1}a)(b^2a^{pt})(b^2a^{pt})^{-1} \ldots$$

$$\ldots (b^2a^{pt})(b^2a^{pt})^{-1}(b^2a)(b^2a^k)(b^2a^k)^{-1} \ldots (b^2a^k)(b^2a^k)^{-1}[b, a^k]$$

with ${(t_2 - 1)}/{2}$ occurrences of the product b^2b^{-2}, with ${t_p}/{2}$ occurrences of the product a^pa^{-p}, with ${(t_{p^2} - 1)}/{2}$ occurrences of the product $a^{-1}a$, ${t_{2p}}/{2}$ occurrences of the product $(b^2a^{pt})(b^2a^{pt})^{-1}$, ${(t_{2p^2} - 1)}/{2}$ occurrences of the product $(b^2a^k)(b^2a^k)^{-1}$ so that,

$$\prod_{i=1}^{t_2}\phi(x_i)\prod_{i=1}^{t_p}\phi(z_i)\prod_{i=1}^{t_{p^2}}\phi(u_i)\prod_{i=1}^{t_{2p}}\phi(v_i)\prod_{i=1}^{t_{2p^2}}\phi(w_i)\prod_{i=1}^{\gamma}[\phi(\alpha_i),\phi(\beta_i)] = 1$$

gives $2k \equiv 1 (\mathrm{mod}\ p^2)$.

Therefore k has a unique solution since $(2, p^2) = 1$.

Hence ϕ is a homomorphism, also since a and b generate M, ϕ is onto. ϕ preserves the order of the finite order generators of Γ, therefore ϕ is a smooth epimorphism.

Note : In the above case if $t_4 \neq 0$ then $\gamma \geq 0$.

In the above mapping in place of the infinite order generators we map the fourth order generators of Γ as follows :

$\phi(y_1) = b$, $\phi(y_2) = b^3a^k$, $0 \leq k < p^2$,

$\phi(y_i) = b^{-1}$, $i = 3, 5, \ldots, t_4 - 1$,

$\phi(y_i) = b$, $i = 4, 6, \ldots, t_4$,

so that ϕ will be a homomorphism and we get $k \equiv 1 (\mathrm{mod}\ p^2)$ which has a solution. Also ϕ preserves the periods of Γ, so ϕ is a smooth epimorphism.

(g) $t_4 = 0$; t_p, t_{2p}, t_{2p^2} all are odd and t_2, t_{p^2} both are even then $\gamma \geq 1$.

$\phi(x_i) = b^2$, $i = 1, 3, \ldots, t_2 - 1$.

$\phi(x_i) = b^{-2}$, $i = 2, 4, \ldots, t_2$.

$\phi(z_1) = a^{py_1}$, $1 < y_1 < p$.

$\phi(z_i) = a^p$, $i = 2, 4, \ldots, t_p - 1$.

$\phi(z_i) = a^{-p}$, $i = 3, 5, \ldots, t_p$.

$\phi(u_i) = a$, , $i = 1, 3, \ldots, t_{p^2} - 1$.

$\phi(u_i) = a^{-1}$, $i = 2, 4, \ldots, t_{p^2}$.

$\phi(v_1) = b^2 a^{p(1-y_1)}$, $1 < y_1 < p$.

$\phi(v_i) = b^2 a^{pt}$, $1 \leq t < p$ and $i = 2, 4, \ldots, t_{2p} - 1$.

$\phi(v_i) = (b^2 a^{pt})^{-1}$, $1 \leq t < p$ and $i = 3, 5, \ldots, t_{2p}$.

$\phi(w_1) = b^2 a$, $\phi(w_i) = b^2 a^k$, $(k, p) = 1$ and $i = 2, 4, \ldots, t_{2p^2} - 1$.

$\phi(w_i) = (b^2 a^k)^{-1}$, $(k, p) = 1$ and $i = 3, 5, \ldots, t_{2p^2}$.

$\phi(\alpha_1) = b$, $\phi(\beta_1) = a^k$, $(k, p) = 1$.

$\phi(\alpha_i) = 1 = \phi(\beta_i)$, $i = 2, 3, \ldots, \gamma$.

Now, $$\prod_{i=1}^{t_2} \phi(x_i)\phi(z_1) \prod_{i=2}^{t_p} \phi(z_i) \prod_{i=1}^{t_{p^2}} \phi(u_i)\phi(v_1) \prod_{i=2}^{t_{2p}} \phi(v_i)\phi(w_1) \prod_{i=2}^{t_{2p^2}} \phi(w_i) \prod_{i=1}^{\gamma} [\phi(\alpha_i), \phi(\beta_i)]$$

$$= (b^2 b^{-2} \ldots b^2 b^{-2})\, a^{py_1} (a^p a^{-p} \ldots a^p a^{-p})(a^{-1} a \ldots a a^{-1})\, b^2 a^{p(1-y_1)} b^2 a^{pt} (b^2 a^{pt})^{-1}$$

$$\ldots (b^2 a^{pt})(b^2 a^{pt})^{-1} b^2 a (b^2 a^k)(b^2 a^k)^{-1} \ldots (b^2 a^k)(b^2 a^k)^{-1} [b, a^k]$$

with ${}^{t_2}\!/_2$ occurrences of the product b^2b^{-2}, with ${}^{t_p - 1}\!/_2$ occurrences of the product a^pa^{-p}, ${}^{t_{p^2}}\!/_2$ occurrences of the product aa^{-1}, ${}^{(t_{2p} - 1)}\!/_2$ occurrences of the product $(b^2a^{pt})(b^2a^{pt})^{-1}$, ${}^{(t_{2p^2} - 1)}\!/_2$ occurrences of the product $(b^2a^k)(b^2a^k)^{-1}$ so that

$$\prod_{i=1}^{t_2}\phi(x_i)\prod_{i=1}^{t_p}\phi(z_i)\prod_{i=1}^{t_{p^2}}\phi(u_i)\prod_{i=1}^{t_{2p}}\phi(v_i)\prod_{i=1}^{t_{2p^2}}\phi(w_i)\prod_{i=1}^{\gamma}[\phi(\alpha_i),\phi(\beta_i)]=1$$

gives $2k \equiv -(p + 1) \pmod{p^2}$.

Therefore k has a unique solution since $(2, p^2) = 1$.

Hence ϕ is a homomorphism, also since a and b generate M, ϕ is onto. ϕ preserves the order of the finite order generators of Γ, therefore ϕ is a smooth epimorphism.

Note : In the above case if $t_4 \neq 0$ then $\gamma \geq 0$.

In the above mapping in place of the infinite order generators we map the fourth order generators of Γ as follows :

$\phi(y_1) = b$, $\phi(y_2) = b^3a^k$,

$\phi(y_i) = b^{-1}$, $i = 3, 5, \ldots, t_4 - 1$,

$\phi(y_i) = b$, $i = 4, 6, \ldots, t_4$,

so that ϕ will be a homomorphism and we get $k \equiv -(p + 1) \pmod{p^2}$ which has a solution.

Also ϕ is a smooth epimorphism since ϕ preserves the periods of Γ.

(h) $t_4 = 0$; t_{p^2}, t_{2p}, t_{2p^2} all are odd and t_2, t_p both are even then $\gamma \geq 1$.

$\phi(x_i) = b^2$, $i = 1, 3, \ldots, t_2 - 1$.

$\phi(x_i) = b^{-2}$, $i = 2, 4, \ldots, t_2$.

$\phi(z_i) = a^p$, $i = 1, 3, \ldots, t_p - 1$.

$\phi(z_i) = a^{-p}$, $i = 2, 4, \ldots, t_p$.

$\phi(u_1) = a^2$, $\phi(u_i) = a^{-1}$, $i = 2, 4, \ldots, t_{p^2} - 1$.

$\phi(u_i) = a$, $i = 3, 5, \ldots, t_{p^2}$.

$\phi(v_1) = b^2a^p$, $\phi(v_i) = b^2a^{pt}$, $1 \leq t < p$ and $i = 2, 4, \ldots, t_{2p} - 1$.

$\phi(v_i) = (b^2a^{pt})^{-1}$, $1 \leq t < p$ and $i = 3, 5, \ldots, t_{2p}$.

$\phi(w_1) = b^2a^{-(p+2)}$, $\phi(w_i) = (b^2a^k)^{-1}$, $i = 2, 4, \ldots, t_{2p^2} - 1$; $(k, p) = 1$.

$\phi(w_i) = b^2a^k$, $i = 3, 5, \ldots, t_{2p^2}$, $(k, p) = 1$.

$\phi(\alpha_1) = b = \phi(\beta_1)$,

$\phi(\alpha_i) = 1 = \phi(\beta_i)$, $i = 2, 3, \ldots, \gamma$

Now, $$\prod_{i=1}^{t_2}\phi(x_i)\prod_{i=1}^{t_p}\phi(z_i)\phi(u_1)\prod_{i=2}^{t_{p^2}}\phi(u_i)\phi(v_1)\prod_{i=2}^{t_{2p}}\phi(v_i)\phi(w_1)\prod_{i=2}^{t_{2p^2}}\phi(w_i)\prod_{i=1}^{\gamma}[\phi(\alpha_i),\phi(\beta_i)]$$

$$= (b^2b^{-2} \ldots b^2b^{-2})(a^pa^{-p} \ldots a^pa^{-p})a^2(a^{-1}a \ldots a^{-1}a)b^2a^p((b^2a^{pt})(b^2a^{pt})^{-1} \ldots$$
$$\ldots (b^2a^{pt})(b^2a^{pt})^{-1})b^2a^{-(p+2)}((b^2a^k)(b^2a^k)^{-1} \ldots (b^2a^k)(b^2a^k)^{-1})[b, b]$$

with $t_2/2$ occurrences of the product b^2b^{-2}, $t_p/2$ occurrences of the product a^pa^{-p}, $(t_{p^2} - 1)/2$ occurrences of the product $a^{-1}a$, $(t_{2p} - 1)/2$ occurrences of the product $(b^2a^{pt})(b^2a^{pt})^{-1}$, $(t_{2p^2} - 1)/2$ occurrences of the product $(b^2a^k)(b^2a^k)^{-1}$ so that

$$\prod_{i=1}^{t_2}\phi(x_i)\prod_{i=1}^{t_p}\phi(z_i)\phi(u_1)\prod_{i=2}^{t_{p^2}}\phi(u_i)\phi(v_1)\prod_{i=2}^{t_{2p}}\phi(v_i)\prod_{i=1}^{t_{2p^2}}\phi(w_i)\prod_{i=1}^{\gamma}[\phi(\alpha_i),\phi(\beta_i)]=1.$$

Therefore ϕ is a homomorphism. Also since a and b generate M, ϕ is onto. ϕ preserves the order of the finite order generators of Γ, therefore ϕ is a smooth epimorphism.

Note : In the above case if $t_4 \neq 0$ then $\gamma \geq 0$.

In the above mapping in place of the infinite order generators we map the fourth order generators of Γ as follows :

$\phi(y_i) = b$, $i = 1, 3, 5, \ldots, t_4 - 1$,

$\phi(y_i) = b^{-1}$, $i = 2, 4, \ldots, t_4$,

so that ϕ will be a homomorphism. Also ϕ preserves the periods of Γ, so ϕ is a smooth epimorphism.

2 (a) $t_4 = 0$; t_2, t_p, t_{p^2}, t_{2p}, t_{2p^2} all are (non-zero) even and $\gamma \geq 1$.

$\phi(x_i) = b^2$, $i = 1, 3, \ldots, t_2 - 1$.

$\phi(x_i) = b^{-2}$, $i = 2, 4, \ldots, t_2$.

$\phi(z_i) = a^{pt}$, $i = 1, 3, \ldots, t_p - 1$ and $1 \leq t < p$.

$\phi(z_i) = a^{-pt}$, $i = 2, 4, \ldots, t_p$ and $1 \leq t < p$.

$\phi(u_i) = a$, $i = 1, 3, \ldots, t_{p^2} - 1$.

$\phi(u_i) = a^{-1}$, $i = 2, 4, \ldots, t_{p^2}$.

$\phi(v_i) = b^2a^{pt}$, $1 \leq t < p$ and $i = 1, 3, \ldots, t_{2p} - 1$.

$\phi(v_i) = (b^2a^{pt})^{-1}$, $1 \leq t < p$ and $i = 2, 4, \ldots, t_{2p}$.

$\phi(w_i) = b^2a^k$, $(k, p) = 1$ and $i = 1, 3, \ldots, t_{2p^2} - 1$.

$\phi(w_i) = (b^2a^k)^{-1}$, $(k, p) = 1$ and $i = 2, 4, \ldots, t_{2p^2}$.

$\phi(\alpha_1) = b = \phi(\beta_1)$.

$\phi(\alpha_i) = 1 = \phi(\beta_i)$, $i = 2, 3, \ldots, \gamma$.

Now, $$\prod_{i=1}^{t_2}\phi(x_i)\prod_{i=1}^{t_p}\phi(z_i)\prod_{i=1}^{t_{p^2}}\phi(u_i)\prod_{i=1}^{t_{2p}}\phi(v_i)\prod_{i=1}^{t_{2p^2}}\phi(w_i)\prod_{i=1}^{\gamma}[\phi(\alpha_i),\phi(\beta_i)]$$

$$= (b^2b^{-2} \ldots b^2b^{-2})(a^{pt}a^{-pt} \ldots a^{pt}a^{-pt})(aa^{-1} \ldots aa^{-1})((b^2a^{pt})(b^2a^{pt})^{-1} \ldots$$
$$\ldots (b^2a^{pt})(b^2a^{pt})^{-1})((b^2a^k)(b^2a^k)^{-1} \ldots (b^2a^k)(b^2a^k)^{-1})[b, b] \,.\, 1$$

with ${}^{t_2}\!/_2$ occurrences of the product b^2b^{-2}, ${}^{t_p}\!/_2$ occurrences of the product $a^{pt}a^{-pt}$, ${}^{t_{p^2}}\!/_2$ occurrences of the product $a^{-1}a$, ${}^{t_{2p}}\!/_2$ occurrences of the product $(b^2a^{pt})(b^2a^{pt})^{-1}$ and

$t_{2p^2}/2$ occurrences of the product $(b^2a^k)(b^2a^k)^{-1}$ so that

$$\prod_{i=1}^{t_2}\phi(x_i)\prod_{i=1}^{t_p}\phi(z_i)\prod_{i=1}^{t_{p^2}}\phi(u_i)\prod_{i=1}^{t_{2p}}\phi(v_i)\prod_{i=1}^{t_{2p^2}}\phi(w_i)\prod_{i=1}^{\gamma}[\phi(\alpha_i),\phi(\beta_i)]=1$$

Therefore ϕ is a homomorphism, also since a and b generate M, ϕ is onto. ϕ preserves the order of the finite order generators of Γ, therefore ϕ is a smooth epimorphism.

Note : In the above case if $t_4 \neq 0$ then $\gamma \geq 0$.

In the above mapping in place of the infinite order generators we map the fourth order generators of Γ as follows :

$\phi(y_i) = b$, $i = 1, 3, 5, ..., t_4 - 1$,

$\phi(y_i) = b^{-1}$, $i = 2, 4, ..., t_4$.

Therefore ϕ is a homomorphism and also a smooth epimorphism since ϕ preserves the periods of Γ.

(a_1) $t_4 = 0$; t_2, t_p, t_{2p} all are non-zero even and t_{p^2}, t_{2p^2} both are simultaneously zero then $\gamma \geq 2$.

In the above homomorphism we can map

$\phi(\alpha_2) = a = \phi(\beta_2)$

$\phi(\alpha_i) = 1 = \phi(\beta_i)$, $i = 3, 4, .., \gamma$,

so that ϕ will be an onto homomorphism.

Note : In the above case when

(1) $t_4 = 2$ and t_{p^2}, t_{2p^2} are simultaneously zero then $\gamma \geq 1$.

We can construct the above homomorphism [2(a)] from Γ onto M by replacing $\phi(u_i)$ and $\phi(w_i)$ by :

$\phi(y_1) = b$, $\phi(y_2) = b^3a$

and the infinite order generators by :

$\phi(\alpha_1) = b$, $\phi(\beta_1) = a^k$, $(k, p) = 1$

$\phi(\alpha_i) = 1 = \phi(\beta_i);\ i = 2, 3, \ldots, \gamma,$

so that ϕ will be a homomorphism. Also ϕ preserves the periods of Γ, so ϕ is a smooth epimorphism.

(2) $t_4 \geq 4$ and t_{p^2}, t_{2p^2} are simultaneously zero then $\gamma \geq 0$.

We can construct the above homomorphism [2(a)] from Γ onto M by replacing $\phi(\alpha_i)$ and $\phi(\beta_i)$ by :

$\phi(y_1) = (ba)^{-1},\ \phi(y_2) = b,\ \phi(y_3) = b,\ \phi(y_4) = b^3a^k,\ (k, p) = 1$

$\phi(y_i) = b,\ i = 5, 7, \ldots, t_4 - 1,$

$\phi(y_i) = b^{-1},\ i = 6, 8, \ldots, t_4,$

so that ϕ will be an onto homomorphism.

(3) $t_4 \geq 4$ and one of t_{p^2}, t_{2p^2} is zero then $\gamma \geq 0$.

We can construct the above homomorphism [2(a)] from Γ onto M by replacing $\phi(\alpha_i)$ and $\phi(\beta_i)$ by :

$\phi(y_1) = b,\ \phi(y_2) = b^{-1},$

$\phi(y_i) = b,\ i = 3, 5, \ldots, t_4 - 1,$

$\phi(y_i) = b^{-1},\ i = 4, 6, \ldots, t_4,$

so that ϕ will be an onto homomorphism.

(b) $t_4 = 0$; t_2, t_{2p} both are odd and t_p, t_{p^2}, t_{2p^2} all are (non-zero) even then $\gamma \geq 1$.

$\phi(x_1) = b^2,\ \phi(x_i) = b^{-2},\ i = 2, 4, \ldots, t_2 - 1.$

$\phi(x_i) = b^2,\ i = 3, 5, \ldots, t_2.$

$\phi(z_i) = a^{pt},\ 1 \leq t < p$ and $i = 1, 3, \ldots, t_p - 1.$

$\phi(z_i) = a^{-pt},\ 1 \leq t < p$ and $i = 2, 4, \ldots, t_p.$

$\phi(u_i) = a,\ i = 1, 3, \ldots, t_{p^2} - 1.$

$\phi(u_i) = a^{-1},\ i = 2, 4, \ldots, t_{p^2}.$

$\phi(v_1) = b^2a^{2p},\ \phi(v_i) = b^2a^{pt},\ 1 \leq t < p$ and $i = 2, 4, \ldots, t_{2p} - 1.$

$\phi(v_i) = (b^2a^{pt})^{-1}$, $1 \leq t < p$ and $i = 3, 5, \ldots, t_{2p}$.

$\phi(w_i) = b^2a^k$, $(k, p) = 1$ and $i = 1, 3, \ldots, t_{2p^2} - 1$.

$\phi(w_i) = (b^2a^k)^{-1}$, $(k, p) = 1$ and $i = 2, 4, \ldots, t_{2p^2}$.

$\phi(\alpha_1) = b$, $\phi(\beta_1) = a^{-p}$.

$\phi(\alpha_i) = 1 = \phi(\beta_i)$, $i = 2, 3, 4, \ldots, \gamma$.

Now, $$\phi(x_1)\prod_{i=2}^{t_2}\phi(x_i)\prod_{i=1}^{t_p}\phi(z_i)\prod_{i=1}^{t_{p^2}}\phi(u_i)\phi(v_1)\prod_{i=2}^{t_{2p}}\phi(v_i)\prod_{i=1}^{t_{2p^2}}\phi(w_i)\prod_{i=1}^{\gamma}[\phi(\alpha_i),\phi(\beta_i)]$$

$$= b^2(b^{-2}b^2 \ldots b^{-2}b^2)(a^{pt}a^{-pt} \ldots a^{pt}a^{-pt})(aa^{-1} \ldots aa^{-1})b^2a^{2p}((b^2a^{pt})(b^2a^{pt})^{-1} \ldots$$
$$\ldots (b^2a^{pt})(b^2a^{pt})^{-1})((b^2a^k)(b^2a^k)^{-1} \ldots (b^2a^k)(b^2a^k)^{-1})[b, a^{-p}] \,.\, 1$$

with ${(t_2 - 1)}/{2}$ occurrences of the product $b^{-2}b^2$, ${t_p}/{2}$ occurrences of the product $a^{pt}a^{-pt}$, ${t_{p^2}}/{2}$ occurrences of the product aa^{-1}, ${(t_{2p} - 1)}/{2}$ occurrences of the product $(b^2a^{pt})(b^2a^{pt})^{-1}$, and ${t_{2p^2}}/{2}$ occurrences of the product $(b^2a^k)(b^2a^k)^{-1}$ so that

$$\prod_{i=1}^{t_2}\phi(x_i)\prod_{i=1}^{t_p}\phi(z_i)\prod_{i=1}^{t_{p^2}}\phi(u_i)\prod_{i=1}^{t_{2p}}\phi(v_i)\prod_{i=1}^{t_{2p^2}}\phi(w_i)\prod_{i=1}^{\gamma}[\phi(\alpha_i),\phi(\beta_i)] = 1.$$

Therefore ϕ is a homomorphism, also since a and b generate M, ϕ is onto. ϕ preserves the order of the finite order generators of Γ, hence ϕ is a smooth epimorphism.

Note : In the above case if $t_4 \neq 0$ then $\gamma \geq 0$.

In the above mapping in place of the infinite order generators we map the fourth order generators of Γ as follows :

$\phi(y_1) = (ba)^{-1}$, $\phi(y_2) = ba^k$, $(k, p) = 1$

$\phi(y_i) = b$, $i = 3, 5, \ldots, t_4 - 1$,

$\phi(y_i) = b^{-1}$, $i = 4, 6, \ldots, t_4$,

so that ϕ will be an onto homomorphism.

(b_1) If $t_4 = 0$; t_2, t_{2p} both are odd and t_p even, t_{p^2}, t_{2p^2} both are simultaneously zero then $\gamma \geq 2$.

In the above homomorphism [2(b)] we can map

$$\phi(\alpha_2) = a = \phi(\beta_2)$$

$$\phi(\alpha_i) = 1 = \phi(\beta_i),\ i = 3, 4, .., \gamma$$

so that ϕ will be an onto homomorphism.

Note : In the above case when

(1) $t_4 = 2$ and t_{p^2}, t_{2p^2} are simultaneously zero then $\gamma \geq 1$.

We can construct the above homomorphism [2(b)] from Γ onto M by replacing $\phi(u_i)$ and $\phi(w_i)$ by :

$$\phi(y_1) = (ba)^{-1},\ \phi(y_2) = b$$

and the infinite order generators by :

$$\phi(\alpha_1) = b,\ \phi(\beta_1) = a^k,\ (k, p) = 1$$

$$\phi(\alpha_i) = 1 = \phi(\beta_i);\ i = 2, 3, ..., \gamma.$$

so that ϕ will be a homomorphism and also a smooth epimorphism, since ϕ preserves the periods of Γ.

(2) $t_4 \geq 4$ and t_{p^2}, t_{2p^2} simultaneously zero then $\gamma \geq 0$.

The above homomorphism [2(b)] can be constructed from Γ onto M by replacing $\phi(u_i)$ and $\phi(w_i)$ by :

$$\phi(y_1) = (ba)^{-1},\ \phi(y_2) = b$$

$$\phi(y_3) = b,\ \phi(y_4) = b^3a^k,\ (0 \leq k < p^2)$$

$$\phi(y_i) = b,\ i = 5, 7, ..., t_4 - 1$$

$$\phi(y_i) = b^{-1},\ i = 6, 8, ..., t_4$$

so that ϕ will be an onto homomorphism.

(3) $t_4 \geq 4$ and one of t_{p^2}, t_{2p^2} is zero then $\gamma \geq 0$.

The fourth order generators can be mapped as follows :

$\phi(y_1) = (ba)^{-1}$, $\phi(y_2) = ba^k$, $(k, p) = 1$

$\phi(y_i) = b$, $i = 3, 5, \ldots, t_4 - 1$

$\phi(y_i) = b^{-1}$, $i = 4, 6, \ldots, t_4$

so that ϕ will be an onto homomorphism.

(c) $t_4 = 0$*;* t_2*,* t_{2p^2} ***both are odd and*** t_p*,* t_{p^2}, t_{2p} ***all are even then*** $\gamma \geq 1$*.*

$\phi(x_1) = b^2$, $\phi(x_i) = b^2$, $i = 2, 4, \ldots, t_2 - 1$.

$\phi(x_i) = b^{-2}$, $i = 3, 5, \ldots, t_2$.

$\phi(z_i) = a^{pt}$, $1 \leq t < p$ and $i = 1, 3, \ldots, t_p - 1$.

$\phi(z_i) = a^{-pt}$, $1 \leq t < p$ and $i = 2, 4, \ldots, t_p$.

$\phi(u_i) = a$, $i = 1, 3, \ldots, t_{p^2} - 1$.

$\phi(u_i) = a^{-1}$, $i = 2, 4, \ldots, t_{p^2}$.

$\phi(v_i) = b^2a^{pt}$, $1 \leq t < p$ and $i = 1, 3, \ldots, t_{2p} - 1$.

$\phi(v_i) = (b^2a^{pt})^{-1}$, $1 \leq t < p$ and $i = 2, 4, \ldots, t_{2p}$.

$\phi(w_1) = b^2a^2$, $\phi(w_i) = b^2a^k$, $(k, p) = 1$ and $i = 2, 4, \ldots, t_{2p^2} - 1$.

$\phi(w_i) = (b^2a^k)^{-1}$, $(k, p) = 1$ and $i = 3, 5, \ldots, t_{2p^2}$.

$\phi(\alpha_1) = a$, $\phi(\beta_1) = b$.

$\phi(\alpha_i) = 1 = \phi(\beta_i)$, $i = 2, 3, \ldots, \gamma$.

Now, $$\phi(x_1)\prod_{i=2}^{t_2}\phi(x_i)\prod_{i=1}^{t_p}\phi(z_i)\prod_{i=1}^{t_{p^2}}\phi(u_i)\prod_{i=1}^{t_{2p}}\phi(v_i)\phi(w_1)\prod_{i=2}^{t_{2p^2}}\phi(w_i)\prod_{i=1}^{\gamma}[\phi(\alpha_i),\phi(\beta_i)]$$

$$= b^2(b^2b^{-2} \ldots b^2b^{-2})(a^{pt}a^{-pt} \ldots a^{pt}a^{-pt})(aa^{-1} \ldots aa^{-1})(b^2a^{pt}(b^2a^{pt})^{-1} \ldots$$

$$\ldots b^2a^{pt}(b^2a^{pt})^{-1})b^2a^2((b^2a^k)(b^2a^k)^{-1} \ldots (b^2a^k)(b^2a^k)^{-1})[a, b] \,.\, 1$$

with $(t_2 - 1)/2$ occurrences of the product b^2b^{-2}, $t_p/2$ occurrences of the product $a^{pt}a^{-pt}$, $t_{p^2}/2$ occurrences of the product aa^{-1}, $t_{2p}/2$ occurrences of the product $(b^2a^{pt})(b^2a^{pt})^{-1}$, and $(t_{2p^2}-1)/2$ occurrences of the product $(b^2a^k)(b^2a^k)^{-1}$ so that

$$\prod_{i=1}^{t_2}\phi(x_i)\prod_{i=1}^{t_p}\phi(z_i)\prod_{i=1}^{t_{p^2}}\phi(u_i)\prod_{i=1}^{t_{2p}}\phi(v_i)\prod_{i=1}^{t_{2p^2}}\phi(w_i)\prod_{i=1}^{\gamma}[\phi(\alpha_i),\phi(\beta_i)]=1.$$

Therefore ϕ is a homomorphism, also since a and b generate M, ϕ is onto. ϕ preserves the order of the finite order generators of Γ, hence ϕ is a smooth epimorphism.

Note : In the above case if $t_4 \neq 0$ then $\gamma \geq 0$.

In the above mapping in place of the infinite order generators we map the fourth order generators of Γ as follows :

$\phi(y_1) = b$, $\phi(y_2) = b^3a^k$, $(k, p) = 1$

$\phi(y_i) = b^{-1}$, $i = 3, 5, \ldots, t_4 - 1$,

$\phi(y_i) = b$, $i = 4, 6, \ldots, t_4$,

so that ϕ will be a homomorphism. Thus getting,

$k + 2 \equiv 0 \pmod{p^2}$

i.e., $k \equiv -2 \pmod{p^2}$, which has a solution.

Hence ϕ is a homomorphism and also a smooth epimorphism, since ϕ preserves the periods of Γ.

(d) $t_4 = 0$; t_p, t_{p^2} both are odd and t_2, t_{2p}, t_{2p^2} all are even then $\gamma \geq 1$.

$\phi(x_i) = b^2$, $i = 1, 3, \ldots, t_2 - 1$.

$\phi(x_i) = b^{-2}$, $i = 2, 4, \ldots, t_2$.

$\phi(z_1) = a^{-p}$, $\phi(z_i) = a^{pt}$, $1 \leq t < p$ and $i = 2, 4, \ldots, t_p - 1$.

$\phi(z_i) = a^{-pt}$, $1 \leq t < p$ and $i = 3, 5, \ldots, t_p$.

$\phi(u_1) = a^2$, $\phi(u_i) = a$, $i = 2, 4, \ldots, t_{p^2} - 1$.

$\phi(u_i) = a^{-1}$, $i = 3, 5, \ldots, t_{p^2}$.

$\phi(v_i) = b^2a^{pt}$, $1 \leq t < p$ and $i = 1, 3, \ldots, t_{2p} - 1$.

$\phi(v_i) = (b^2a^{pt})^{-1}$, $1 \leq t < p$ and $i = 2, 4, \ldots, t_{2p}$.

$\phi(w_i) = b^2a^k$, $(k, p) = 1$ and $i = 1, 3, \ldots, t_{2p^2} - 1$.

$\phi(w_i) = (b^2a^k)^{-1}$, $(k, p) = 1$ and $i = 2, 4, \ldots, t_{2p^2}$.

$\phi(\alpha_1) = b$, $\phi(\beta_1) = a^k$, $(k, p) = 1$.

$\phi(\alpha_i) = 1 = \phi(\beta_i)$, $i = 2, 3, \ldots, \gamma$.

Now, $$\prod_{i=1}^{t_2}\phi(x_i)\phi(z_1)\prod_{i=2}^{t_p}\phi(z_i)\phi(u_1)\prod_{i=2}^{t_{p^2}}\phi(u_i)\prod_{i=1}^{t_{2p}}\phi(v_i)\prod_{i=1}^{t_{2p^2}}\phi(w_i)\prod_{i=1}^{\gamma}[\phi(\alpha_i),\phi(\beta_i)]$$

$$= (b^2b^{-2} \ldots b^2b^{-2})a^{-p}(a^{pt}a^{-pt} \ldots a^{pt}a^{-pt})a^2(aa^{-1} \ldots aa^{-1})(b^2a^{pt}(b^2a^{pt})^{-1} \ldots$$
$$\ldots b^2a^{pt}(b^2a^{pt})^{-1})(b^2a^k(b^2a^k)^{-1} \ldots b^2a^k(b^2a^k)^{-1})[b, a^k] \,.\, 1$$

with $t_2/2$ occurrences of the product b^2b^{-2}, $(t_p - 1)/2$ occurrences of the product $a^{pt}a^{-pt}$, $(t_{p^2} - 1)/2$ occurrences of the product aa^{-1}, $t_{2p}/2$ occurrences of the product $(b^2a^{pt})(b^2a^{pt})^{-1}$ and $t_{2p^2}/2$ occurrences of the product $(b^2a^k)(b^2a^k)^{-1}$ so that,

$$\prod_{i=1}^{t_2}\phi(x_i)\prod_{i=1}^{t_p}\phi(z_i)\prod_{i=1}^{t_{p^2}}\phi(u_i)\prod_{i=1}^{t_{2p}}\phi(v_i)\prod_{i=1}^{t_{2p^2}}\phi(w_i)\prod_{i=1}^{\gamma}[\phi(\alpha_i),\phi(\beta_i)] = 1$$

gives $2k \equiv p - 2 \pmod{p^2}$ which has a solution as $(2, p^2) = 1$.

Therefore ϕ is a homomorphism and also a smooth epimorphism, since ϕ preserves the order of the finite order generators of Γ.

Note : In the above case if $t_4 \neq 0$ then $\gamma \geq 0$.

In the above mapping in place of the infinite order generators we map the fourth order generators of Γ as follows :

$\phi(y_1) = b$, $\phi(y_2) = b^3a^k$, $0 \leq k < p^2$,

$\phi(y_i) = b$, $i = 3, 5, \ldots, t_4 - 1$,

$\phi(y_i) = b^{-1}$, $i = 4, 6, \ldots, t_4$,

so that ϕ will be a homomorphism. Thus getting $k \equiv p - 2 \pmod{p^2}$, which has a solution.

Hence ϕ is a homomorphism and also smooth epimorphism, since ϕ preserves the periods of Γ.

(e) $t_4 = 0$; t_{2p}, t_{2p^2} *both are odd and* t_2, t_p, t_{p^2} *all are even then* $\gamma \geq 1$.

$\phi(x_i) = b^2$, $i = 1, 3, \ldots, t_2 - 1$.

$\phi(x_i) = b^{-2}$, $i = 2, 4, \ldots, t_2$.

$\phi(z_i) = a^{pt}$, $1 \leq t < p$ and $i = 1, 3, \ldots, t_p - 1$.

$\phi(z_i) = a^{-pt}$, $1 \leq t < p$ and $i = 2, 4, \ldots, t_p$.

$\phi(u_i) = a$, $i = 1, 3, \ldots, t_{p^2} - 1$.

$\phi(u_i) = a^{-1}$, $i = 2, 4, \ldots, t_{p^2}$.

$\phi(v_1) = b^2a^p$, $\phi(v_i) = b^2a^{pt}$, $1 \leq t < p$ and $i = 2, 4, \ldots, t_{2p} - 1$.

$\phi(v_i) = (b^2a^{pt})^{-1}$, $1 \leq t < p$ and $i = 3, 5, \ldots, t_{2p}$.

$\phi(w_1) = b^2a^{-(p+1)}$, $\phi(w_i) = b^2a^k$, $(k, p) = 1$ and $i = 2, 4, \ldots, t_{2p^2} - 1$.

$\phi(w_i) = (b^2a^k)^{-1}$, $(k, p) = 1$ and $i = 3, 5, \ldots, t_{2p^2}$.

$\phi(\alpha_1) = b$, $\phi(\beta_1) = a^k$.

$\phi(\alpha_i) = 1 = \phi(\beta_i)$, $i = 2, 3, \ldots, \gamma$.

Now, $$\prod_{i=1}^{t_2}\phi(x_i)\prod_{i=1}^{t_p}\phi(z_i)\prod_{i=1}^{t_{p^2}}\phi(u_i)\phi(v_1)\prod_{i=2}^{t_{2p}}\phi(v_i)\phi(w_1)\prod_{i=2}^{t_{2p^2}}\phi(w_i)\prod_{i=1}^{\gamma}[\phi(\alpha_i),\phi(\beta_i)]$$

$$= (b^2b^{-2} \ldots b^2b^{-2})(a^{pt}a^{-pt} \ldots a^{pt}a^{-pt})(aa^{-1} \ldots aa^{-1})b^2a^p((b^2a^{pt})(b^2a^{pt})^{-1} \ldots$$
$$\ldots (b^2a^{pt})(b^2a^{pt})^{-1})b^2a^{-(p+1)}(b^2a^k(b^2a^k)^{-1} \ldots b^2a^k(b^2a^k)^{-1})[b, a^k]$$

with $t_2/2$ occurrences of the product b^2b^{-2}, $t_p/2$ occurrences of the product $a^{pt}a^{-pt}$, $t_{p^2}/2$ occurrences of the product aa^{-1}, $(t_{2p}-1)/2$ occurrences of the product $b^2a^{pt}(b^2a^{pt})^{-1}$ and $(t_{2p^2}-1)/2$ occurrences of the product $b^2a^k(b^2a^k)^{-1}$ so that

$$\prod_{i=1}^{t_2}\phi(x_i)\prod_{i=1}^{t_p}\phi(z_i)\prod_{i=1}^{t_{p^2}}\phi(u_i)\prod_{i=1}^{t_{2p}}\phi(v_i)\prod_{i=1}^{t_{2p^2}}\phi(w_i)\prod_{i=1}^{\gamma}[\phi(\alpha_i),\phi(\beta_i)]=1$$

gives $2k \equiv 1 \pmod{p^2}$ which has a unique solution.

Therefore ϕ is a homomorphism and also a smooth epimorphism, since ϕ preserves the order of the finite order generators of Γ.

Note : In the above case if $t_4 \neq 0$ then $\gamma \geq 0$.

In the above mapping in place of the infinite order generators we map the fourth order generators of Γ as follows :

$\phi(y_1) = b$, $\phi(y_2) = b^3a^k$,

$\phi(y_i) = b$, $i = 3, 5, \ldots, t_4 - 1$,

$\phi(y_i) = b^{-1}$, $i = 4, 6, \ldots, t_4$,

so that ϕ will be a homomorphism. Thus getting $k \equiv 1 \pmod{p^2}$, which has a solution.

Hence ϕ is a homomorphism and also a smooth epimorphism, since ϕ preserves the periods of Γ.

(f) $t_4 = 0$; t_2, t_p, t_{p^2}, t_{2p} all are odd and t_{2p^2} even then $\gamma \geq 1$.

$\phi(x_1) = b^2$, $\phi(x_i) = b^{-2}$, $i = 2, 4, \ldots, t_2 - 1$.

$\phi(x_i) = b^2$, $i = 3, 5, \ldots, t_2$.

$\phi(z_1) = a^{py_1}$, $1 < y_1 < p$, $\phi(z_i) = a^{pt}$, $1 \leq t < p$ and $i = 2, 4, \ldots, t_p - 1$.

$\phi(z_i) = (a^{pt})^{-1}$, $1 \leq t < p$ and $i = 3, 5, \ldots, t_p$.

$\phi(u_1) = a^{-2}$, $\phi(u_i) = a$, $i = 2, 4, \ldots, t_{p^2} - 1$.

$\phi(u_i) = a^{-1}$, $i = 3, 5, \ldots, t_{p^2}$.

$\phi(v_1) = b^2 a^{-py_1}$, $1 < y_1 < p$.

$\phi(v_i) = b^2a^{pt}$, $1 \leq t < p$ and $i = 2, 4, \ldots, t_{2p} - 1$.

$\phi(v_i) = (b^2a^{pt})^{-1}$, $1 \leq t < p$ and $i = 3, 5, \ldots, t_{2p}$.

$\phi(w_i) = b^2a^k$, $(k, p) = 1$ and $i = 1, 3, \ldots, t_{2p^2} - 1$.

$\phi(w_i) = (b^2a^k)^{-1}$, $(k, p) = 1$ and $i = 2, 4, \ldots, t_{2p^2}$.

$\phi(\alpha_1) = b$, $\phi(\beta_1) = a$.

$\phi(\alpha_i) = 1 = \phi(\beta_i)$, $i = 2, 3, \ldots, \gamma$.

Now, $$\phi(x_1)\prod_{i=2}^{t_2}\phi(x_i)\phi(z_1)\prod_{i=2}^{t_p}\phi(z_i)\phi(u_1)\prod_{i=2}^{t_{p^2}}\phi(u_1)\phi(v_1)\prod_{i=2}^{t_{2p}}\phi(v_i)\prod_{i=1}^{t_{2p^2}}\phi(w_i)\prod_{i=1}^{\gamma}[\phi(\alpha_i),\phi(\beta_i)]$$

$$= b^2(b^2b^{-2} \ldots b^2b^{-2})\,a^{py_1}(a^{pt}a^{-pt} \ldots a^{pt}a^{-pt})a^{-2}(aa^{-1} \ldots aa^{-1})\,b^2a^{-py_1}$$

$$((b^2a^{pt})(b^2a^{pt})^{-1} \ldots (b^2a^{pt})(b^2a^{pt})^{-1})(b^2a^k(b^2a^k)^{-1} \ldots b^2a^k(b^2a^k)^{-1})[b, a].1$$

with ${(t_2 - 1)}/{2}$ occurrences of the product b^2b^{-2}, ${(t_p - 1)}/{2}$ occurrences of the product $a^{pt}(a^{pt})^{-1}$, ${(t_{p^2} - 1)}/{2}$ occurrences of the product aa^{-1}, ${(t_{2p} - 1)}/{2}$ occurrences of the product $b^2a^{pt}(b^2a^{pt})^{-1}$ and ${t_{2p^2}}/{2}$ occurrences of the product $b^2a^k(b^2a^k)^{-1}$ so that,

$$\prod_{i=1}^{t_2}\phi(x_i)\phi(z_1)\prod_{i=2}^{t_p}\phi(z_i)\prod_{i=1}^{t_{p^2}}\phi(u_i)\prod_{i=1}^{t_{2p}}\phi(v_i)\prod_{i=1}^{t_{2p^2}}\phi(w_i)\prod_{i=1}^{\gamma}[\phi(\alpha_i),\phi(\beta_i)] = 1$$

Therefore ϕ is a homomorphism, also since a and b generate M, ϕ is onto. ϕ preserves the order of the finite order generators of Γ, hence ϕ is a smooth epimorphism.

Note : In the above case if $t_4 \neq 0$ then $\gamma \geq 0$.

In the above mapping in place of the infinite order generators we map the fourth order generators of Γ as follows :

$\phi(y_1) = b$, $\phi(y_2) = b^3a^k$, $0 \leq k < p^2$

$\phi(y_i) = b$, $i = 3, 5, \ldots, t_4 - 1$,

$\phi(y_i) = b^{-1}$, $i = 4, 6, \ldots, t_4$,

so that ϕ will be a homomorphism. Thus getting $k \equiv 2 \pmod{p^2}$, which has a solution.

Therefore ϕ is a homomorphism and also smooth epimorphism, since ϕ preserves the periods of Γ.

(g) $t_4 = 0$; t_2, t_p, t_{p^2}, t_{2p^2}, *all are odd and* t_{2p} *even then* $\gamma \geq 1$.

$\phi(x_1) = b^2$, $\phi(x_i) = b^{-2}$, $i = 2, 4, \ldots, t_2 - 1$.

$\phi(x_i) = b^2$, $i = 3, 5, \ldots, t_2$.

$\phi(z_1) = a^p$, $\phi(z_i) = a^{pt}$, $1 \leq t < p$ and $i = 2, 4, \ldots, t_p - 1$.

$\phi(z_i) = a^{-pt}$, $1 \leq t < p$ and $i = 3, 5, \ldots, t_p$.

$\phi(u_1) = a^2$, $\phi(u_i) = a$, $i = 2, 4, 6, \ldots, t_{p^2} - 1$.

$\phi(u_i) = a^{-1}$, $i = 3, 5, \ldots, t_{p^2}$.

$\phi(v_i) = b^2a^{pt}$, $1 \leq t < p$ and $i = 1, 3, \ldots, t_{2p} - 1$.

$\phi(v_i) = (b^2a^{pt})^{-1}$, $1 \leq t < p$ and $i = 2, 4, \ldots, t_{2p}$.

$\phi(w_1) = b^2a^{-(p+2)}$, $\phi(w_i) = b^2a^k$, $(k, p) = 1$ and $i = 2, 4, \ldots, t_{2p^2} - 1$.

$\phi(w_i) = (b^2a^k)^{-1}$, $(k, p) = 1$ and $i = 3, 5, \ldots, t_{2p^2}$.

$\phi(\alpha_1) = b = \phi(\beta_1)$.

$\phi(\alpha_i) = 1 = \phi(\beta_i)$, $i = 2, 3, \ldots, \gamma$.

Now, $\phi(x_1)\prod_{i=2}^{t_2}\phi(x_i)\phi(z_1)\prod_{i=2}^{t_p}\phi(z_i)\phi(u_1)\prod_{i=2}^{t_{p^2}}\phi(u_i)\prod_{i=1}^{t_{2p}}\phi(v_i)\phi(w_1)\prod_{i=2}^{t_{2p^2}}\phi(w_i)\prod_{i=1}^{\gamma}[\phi(\alpha_i),\phi(\beta_i)]$

$$= b^2(b^{-2}b^2 \ldots b^{-2}b^2)a^p(a^{pt}a^{-pt} \ldots a^{pt}a^{-pt})a^2(aa^{-1} \ldots aa^{-1})((b^2a^{pt})(b^2a^{pt})^{-1} \ldots$$
$$\ldots (b^2a^{pt})(b^2a^{pt})^{-1})b^2a^{-(p+2)}(b^2a^k(b^2a^k)^{-1} \ldots b^2a^k(b^2a^k)^{-1})[b, b]$$

with $(t_2 - 1)/2$ occurrences of the product $b^{-2}b^2$, $(t_p - 1)/2$ occurrences of the product $a^{pt}(a^{pt})^{-1}$, $(t_{p^2} - 1)/2$ occurrences of the product aa^{-1}, $t_{2p}/2$ occurrences of the product $b^2a^{pt}(b^2a^{pt})^{-1}$ and $(t_{2p^2} - 1)/2$ occurrences of the product $b^2a^k(b^2a^k)^{-1}$ so that,

$$\prod_{i=1}^{t_2}\phi(x_i)\prod_{i=1}^{t_p}\phi(z_i)\prod_{i=1}^{t_{p^2}}\phi(u_i)\prod_{i=1}^{t_{2p}}\phi(v_i)\prod_{i=1}^{t_{2p^2}}\phi(w_i)\prod_{i=1}^{\gamma}[\phi(\alpha_i),\phi(\beta_i)]=1.$$

Therefore ϕ is a homomorphism and also a smooth epimorphism, since ϕ preserves the order of the finite order generators of Γ.

Note : In the above case if $t_4 \neq 0$ then $\gamma \geq 0$.

In the above mapping in place of the infinite order generators we map the fourth order generators of Γ as follows :

$\phi(y_i) = b$, $i = 1, 3, 5, \ldots, t_4 - 1$,

$\phi(y_i) = b^{-1}$, $i = 2, 4, \ldots, t_4$,

so that ϕ will be a homomorphism and also smooth epimorphism, since ϕ preserves the order of the finite order generators of Γ.

(h) $t_4 = 0$; t_p, t_{p^2}, t_{2p}, t_{2p^2} all are odd and t_2 even then $\gamma \geq 1$.

$\phi(x_i) = b^2$, $i = 1, 3, \ldots, t_2 - 1$.

$\phi(x_i) = b^{-2}$, $i = 2, 4, \ldots, t_2$.

$\phi(z_1) = a^{py_1}$, $1 < y_1 < p$.

$\phi(z_i) = a^{pt}$, $1 \leq t < p$ and $i = 2, 4, \ldots, t_p - 1$.

$\phi(z_i) = a^{-pt}$, $1 \leq t < p$ and $i = 3, 5, \ldots, t_p$.

$\phi(u_1) = a^{-2}$, $\phi(u_i) = a$, $i = 2, 4, 6, \ldots, t_{p^2} - 1$.

$\phi(u_i) = a^{-1}$, $i = 3, 5, \ldots, t_{p^2}$.

$\phi(v_1) = b^2a^{-py_1}$, $1 < y_1 < p$.

$\phi(v_i) = b^2a^{pt}$, $1 \leq t < p$ and $i = 2, 4, \ldots, t_{2p} - 1$.

$\phi(v_i) = (b^2a^{pt})^{-1}$, $1 \leq t < p$ and $i = 3, 5, \ldots, t_{2p}$.

$\phi(w_1) = b^2a^2$, $\phi(w_i) = b^2a^k$, $(k, p) = 1$ and $i = 2, 4, \ldots, t_{2p^2} - 1$.

$\phi(w_i) = (b^2a^k)^{-1}$, $(k, p) = 1$ and $i = 3, 5, \ldots, t_{2p^2}$.

- $\phi(\alpha_1) = b = \phi(\beta_1)$.

$\phi(\alpha_i) = 1 = \phi(\beta_i)$, $i = 2, 3, \ldots, \gamma$.

Now, $$\prod_{i=1}^{t_2}\phi(x_i)\phi(z_1)\prod_{i=2}^{t_p}\phi(z_i)\phi(u_1)\prod_{i=2}^{t_{p^2}}\phi(u_i)\phi(v_1)\prod_{i=2}^{t_{2p}}\phi(v_i)\phi(w_1)\prod_{i=2}^{t_{2p^2}}\phi(w_i)\prod_{i=1}^{\gamma}[\phi(\alpha_i),\phi(\beta_i)]$$

$$= (b^2b^{-2} \ldots b^2b^{-2})\, a^{py_1} (a^{pt}a^{-pt} \ldots a^{pt}a^{-pt})a^{-2}(aa^{-1} \ldots aa^{-1})\, b^2a^{-py_1} (b^2a^{pt}(b^2a^{pt})^{-1}$$

$$\ldots (b^2a^{pt})(b^2a^{pt})^{-1})b^2a^2(b^2a^k(b^2a^k)^{-1} \ldots b^2a^k(b^2a^k)^{-1})[b, b]$$

with ${t_2}/{2}$ occurrences of the product b^2b^{-2}, ${(t_p - 1)}/{2}$ occurrences of the product $a^{pt}(a^{-pt})$, ${(t_{p^2} - 1)}/{2}$ occurrences of the product aa^{-1}, ${(t_{2p} - 1)}/{2}$ occurrences of the product $b^2a^{pt}(b^2a^{pt})^{-1}$, ${(t_{2p^2} - 1)}/{2}$ occurrences of the product $b^2a^k(b^2a^k)^{-1}$ so that,

$$\prod_{i=1}^{t_2}\phi(x_i)\prod_{i=1}^{t_p}\phi(z_i)\prod_{i=1}^{t_{p^2}}\phi(u_i)\prod_{i=1}^{t_{2p}}\phi(v_i)\prod_{i=1}^{t_{2p^2}}\phi(w_i)\prod_{i=1}^{\gamma}[\phi(\alpha_i),\phi(\beta_i)] = 1.$$

Therefore ϕ is a homomorphism and also a smooth epimorphism, since ϕ preserves the order of the finite order generators of Γ.

Note : In the above case if $t_4 \neq 0$ then $\gamma \geq 0$.

In the above mapping in place of the infinite order generators we map the fourth order generators of Γ as follows :

$\phi(y_i) = b$, $i = 1, 3, \ldots, t_4 - 1$,

$\phi(y_i) = b^{-1}$, $i = 2, 4, \ldots, t_4$,

so that ϕ will be a homomorphism and also smooth epimorphism, since ϕ preserves the order of the finite order generators of Γ.

This completes the proof of the sufficiency of the conditions.

3.2 Determination of Minimum genus

In this section we determine the minimum genus of a Riemann surface admitting M as an automorphism group and also find the signature of the corresponding Fuchsian group of which M is a smooth quotient.

Let g be the genus of a compact Riemann surface having M as its automorphism group. Then M is a smooth quotient of some Fuchsian group Γ. We have already proved that Γ must have a signature of the form :

$$(\gamma;\ 2, \ldots, 2, 4, \ldots, 4, p, \ldots, p, p^2, \ldots, p^2, 2p, \ldots, 2p, 2p^2, \ldots, 2p^2)$$

with t_2-occurrences of 2, t_4 occurrences of 4, t_p occurrences of p, t_{p^2} occurrences of p^2, t_{2p} occurrences of 2p and t_{2p^2} occurrences of $2p^2$ where γ is the genus of the Fuchsian group Γ and t_2, t_4, t_p, t_{p^2}, t_{2p}, t_{2p^2} are respectively the number of generators of order 2, 4, p, p^2, 2p and $2p^2$. Also γ, t_2, t_4, ..., t_{2p^2} must satisfy the conditions of the theorem (3.1.1). From (3.4), it follows that the minimum of g is obtained when the right side of (3.4) is minimum.

From (3.4) we get,

$$2(g-1) = 4p^2\left\{2(\gamma-1) + t_2\left(1-\frac{1}{2}\right) + t_4\left(1-\frac{1}{4}\right) + t_p\left(1-\frac{1}{p}\right)\right.$$

$$\left. + t_{p^2}\left(1-\frac{1}{p^2}\right) + t_{2p}\left(1-\frac{1}{2p}\right) + t_{2p^2}\left(1-\frac{1}{2p^2}\right)\right\}. \quad \ldots\ldots(3.2.1)$$

We now examine for what value or values of γ, t_2, t_4, t_p, t_{p^2}, t_{2p} and t_{2p^2} the minimum value of g is obtained.

I. when s = 0 i.e., $t_2 + t_4 + t_p + t_{p^2} + t_{2p} + t_{2p^2} = 0,\ \gamma \geq 2$.

From (3.2.1) the minimum value of g is $4p^2 + 1$.

II. when s = 1 i.e., $t_2 + t_4 + t_p + t_{p^2} + t_{2p} + t_{2p^2} = 1,\ \gamma \geq 1$.

From (3.2.1) we have,

$$2(g-1)=4p^2\left\{\ 2(\gamma-1)+\left(t_2+t_4+t_p+t_{p^2}+t_{2p}+t_{2p^2}\right)\right.$$
$$\left.-\left(\frac{t_2}{2}+\frac{t_4}{4}+\frac{t_p}{p}+\frac{t_{p^2}}{p^2}+\frac{t_{2p}}{2p}+\frac{t_{2p^2}}{2p^2}\right)\right\}.$$

So, $2(g-1)\geq 4p^2\left\{1-\frac{1}{n}\right\}$

vhere n can takes value either p or p^2 since t_2, t_4, t_{2p}, t_{2p^2} cannot take odd values ndividually.

i.e., $g\geq 2p^2\left\{1-\frac{1}{n}\right\}+1$. ......(3.2.2)

The right hand side of (3.2.2) is minimum if $\left\{1-\frac{1}{n}\right\}$ is minimum. Now $1-\frac{1}{n}$ will e minimum when n = p. Therefore in this case the minimum value of g is

$2p(p - 1) + 1$ i.e., $2p^2 - 2p + 1$.

III. when s = 2 i.e. $t_2 + t_4 + t_p + t_{p^2} + t_{2p} + t_{2p^2} = 2$, $\gamma \geq 1$.

From (3.2.1) we have

$$2(g-1)=4p^2\left\{\ 2(\gamma-1)+\left(t_2+t_4+t_p+t_{p^2}+t_{2p}+t_{2p^2}\right)\right.$$
$$\left.-\left(\frac{t_2}{2}+\frac{t_4}{4}+\frac{t_p}{p}+\frac{t_{p^2}}{p^2}+\frac{t_{2p}}{2p}+\frac{t_{2p^2}}{2p^2}\right)\right\}$$

i.e., $g-1\geq 2p^2\left\{2-\left(\frac{t_i}{i}+\frac{t_j}{j}\right)\right\}$. ......(3.2.3)

Since t_4 is always even, so either $t_4 = 0$ or $t_4 \neq 0$.

(i) If $t_4 \neq 0$ then $t_4 = 2$ and other periods are zero.

(ii) If $t_4 = 0$ then by theorem (3.1.1), (i, j) can take the following values :

(2, 2), (p, p), (p^2, p^2), (2p, 2p), $(2p^2, 2p^2)$, (2, 2p), $(2, 2p^2)$, $(2p, 2p^2)$ and (p, p^2).

Right hand side of (3.2.3) will be minimum only when $\frac{t_i}{i}+\frac{t_j}{j}$ is maximum.

This will be maximum when i = j = 2.

Hence $$g-1 \geq 2p^2\left\{2-\left(\frac{1}{2}+\frac{1}{2}\right)\right\}$$

i.e., $g \geq 2p^2 + 1$.

In this case the minimum value of g is $2p^2 + 1$.

IV. when s = 3 i.e., $t_2 + t_4 + t_p + t_{p^2} + t_{2p} + t_{2p^2} = 3$, $\gamma \geq 0$.

From (3.2.1) we have,

$$2(g-1) \geq 4p^2\left\{-2+3-\left(\frac{t_2}{2}+\frac{t_4}{4}+\frac{t_p}{p}+\frac{t_{p^2}}{p^2}+\frac{t_{2p}}{2p}+\frac{t_{2p^2}}{2p^2}\right)\right\}. \quad \text{......(3.2.4)}$$

For the existence of a smooth epimorphism from a Fuchsian group Γ to M, a necessary condition for $\gamma = 0$ is $t_4 \neq 0$. Since t_4 is always even, so for s = 3, $t_4 = 2$ and the other period must be either p or p^2 since t_i can not take odd values for the following values of i :

i = 2, 2p and $2p^2$.

Therefore from (3.2.4) we have

$$2(g-1) \geq 4p^2\left\{1-\left(\frac{t_4}{4}+\frac{t_i}{i}\right)\right\}. \quad \text{......(3.2.5)}$$

The right hand side of (3.2.5) will be minimum only when $t_4 = 2$, i = p.

In this case the minimum value of g is

$(p - 1)^2$ or $p^2 - 2p + 1$.

From the above discussion, we have minimum values of g for small values of

Out of the above possible minimum values of g we can easily come to the conclusion that $p^2 - 2p + 1$ or $(p - 1)^2$ is the least value that g can have so far and the corresponding signature of Γ is $(4, 4, p)$.

We now show that g cannot have a lower minimum for $s \geq 4$.

i.e., $t_2 + t_4 + t_p + t_{p^2} + t_{2p} + t_{2p^2} \geq 4$.

If possible, suppose there is a smaller value of g for some t_2, t_4, t_p, t_{p^2}, t_{2p} and t_{2p^2}.

From (3.2.1) we have

$$2(g-1) \geq -2 + \left(t_2 + t_4 + t_p + t_{p^2} + t_{2p} + t_{2p^2}\right) - \frac{t_2 + t_4 + t_p + t_{p^2} + t_{2p} + t_{2p^2}}{2}$$

$$= \frac{1}{2}\left(t_2 + t_4 + t_p + t_{p^2} + t_{2p} + t_{2p^2}\right) - 2.$$

We are to show that,

$$\frac{t_2 + t_4 + t_p + t_{p^2} + t_{2p} + t_{2p^2}}{2} - 2 < 1 - \frac{1}{2} - \frac{1}{p} \qquad \text{......(3.2.6)}$$

or
$$\frac{t_2 + t_4 + t_p + t_{p^2} + t_{2p} + t_{2p^2}}{2} < 3 - \frac{1}{2} - \frac{1}{p}.$$

Now for $s \geq 6$, left hand side is greater than the right hand side, so we check the minimum value of g for $s = 4$ and $s = 5$.

Consider $s = 4$ i.e., $t_2 + t_4 + t_p + t_{p^2} + t_{2p} + t_{2p^2} = 4$.

From (3.2.1) we get,

$$\frac{2(g-1)}{4p^2} = 2(\gamma - 1) + \left(t_2 + t_4 + t_p + t_{p^2} + t_{2p} + t_{2p^2}\right)$$

$$- \left(\frac{t_2}{2} + \frac{t_4}{4} + \frac{t_p}{p} + \frac{t_{p^2}}{p^2} + \frac{t_{2p}}{2p} + \frac{t_{2p^2}}{2p^2}\right).$$

Now we must have,

$$2(\gamma-1)+\left(t_2+t_4+t_p+t_{p^2}+t_{2p}+t_{2p^2}\right)$$

$$-\left(\frac{t_2}{2}+\frac{t_4}{4}+\frac{t_p}{p}+\frac{t_{p^2}}{p^2}+\frac{t_{2p}}{2p}+\frac{t_{2p^2}}{2p^2}\right)<1-\frac{1}{2}-\frac{1}{p} \qquad \text{......(3.2.7)}$$

because the lower bound of g for s = 3 is obtained when

$$\frac{2(g-1)}{4p^2}=1-\frac{1}{2}-\frac{1}{p} \text{ and } \gamma = 0.$$

If $\gamma = 1$ then (3.2.7) doesnot hold.

We have, $$\frac{t_2}{2}+\frac{t_4}{4}+\frac{t_p}{p}+\frac{t_{p^2}}{p^2}+\frac{t_{2p}}{2p}+\frac{t_{2p^2}}{2p^2}<\frac{1}{2}\left[t_2+t_4+t_p+t_{p^2}+t_{2p}+t_{2p^2}\right]$$

Now, $$2(\gamma-1)+\left(t_2+t_4+t_p+t_{p^2}+t_{2p}+t_{2p^2}\right)-\left(\frac{t_2}{2}+\frac{t_4}{4}+\frac{t_p}{p}+\frac{t_{p^2}}{p^2}+\frac{t_{2p}}{2p}+\frac{t_{2p^2}}{2p^2}\right)$$

$$>2(\gamma-1)+\left(t_2+t_4+t_p+t_{p^2}+t_{2p}+t_{2p^2}\right)-\frac{1}{2}\left[t_2+t_4+t_p+t_{p^2}+t_{2p}+t_{2p^2}\right]$$

$$=2(\gamma-1)+\frac{1}{2}\left[t_2+t_4+t_p+t_{p^2}+t_{2p}+t_{2p^2}\right].$$

Therefore,

$$2(\gamma-1)+\frac{1}{2}\left[t_2+t_4+t_p+t_{p^2}+t_{2p}+t_{2p^2}\right]>2(\gamma-1)+2$$

$= 2[(\gamma - 1) + 1]$ (since s = 4).

Hence the only possibility for getting a lower minimum is $\gamma = 0$.

So, $$2(g-1)\geq 4p^2\left\{-2+4-\left(\frac{t_2}{2}+\frac{t_4}{4}+\frac{t_p}{p}+\frac{t_{p^2}}{p^2}+\frac{t_{2p}}{2p}+\frac{t_{2p^2}}{2p^2}\right)\right\}. \qquad \text{......(3.2.8)}$$

For the existence of a smooth epimorphism from a Fuchsian group Γ to M a necessary condition for $\gamma = 0$ is $t_4 \neq 0$. Since t_4 is always even, so for $s = 4$ we have either $t_4 = 4$ or $t_4 = 2$ and the other periods must be i and j where i and j can take values from the set $\{2, p, p^2, 2p, 2p^2\}$.

If (i) $t_4 = 4$ then t_i, t_j are zero.

(ii) $t_4 = 2$ then i, j may take values from the set $\{2, p, p^2, 2p, 2p^2\}$.

Since the periods of Γ satisfy the conditions of theorem (3.1.1), so i and j can only take the following values that is (i, j) can be one of the following :

$(2, 2)$, (p, p), (p^2, p^2), $(2p, 2p)$, $(2p^2, 2p^2)$, $(2, 2p)$, $(2, 2p^2)$, $(2p, 2p^2)$ and (p, p^2).

From the relation (3.2.8) we get,

$$g-1 \geq 2p^2\left\{2-\left(\frac{t_4}{4}+\frac{t_i}{i}+\frac{t_j}{j}\right)\right\}. \qquad \text{......(3.2.9)}$$

If $t_4 = 4$ then $g \geq 2p^2 + 1$.

If $t_4 = 2$ then from (3.2.9) we have

$$g-1 \geq 2p^2\left\{\frac{3}{2}-\left(\frac{t_i}{i}+\frac{t_j}{j}\right)\right\}.$$

Now right hand side will be minimum when $\left(\frac{t_i}{i}+\frac{t_j}{j}\right)$ is maximum. This will be maximum when $i = 2$, $j = 2$.

Therefore $\quad \frac{t_i}{i}+\frac{t_j}{j}=\frac{1}{2}+\frac{1}{2}=1.$

Hence $\quad g \geq 2p^2\left\{\frac{3}{2}-1\right\}+1$

$= p^2 + 1.$

So in this case the minimum value of g is $p^2 + 1$.

We have seen that this minimum value of g is greater than $p^2 - 2p + 1$. Therefore the minimum value for a compact Riemann surface admitting M as an automorphism group is $p^2 - 2p + 1$ and in that case M is a smooth quotient of a Fuchsian group Γ having the signature (4, 4, p).

We shall also show that there is no minimum value for s = 5.

i.e., $t_2 + t_4 + t_p + t_{p^2} + t_{2p} + t_{2p^2} = 5.$

In this case also the value of g could be less than that of the case s = 3 only when $\gamma = 0$.

For s = 5, $\gamma \geq 0$.

So, $$2(g-1) \geq 4p^2\left\{-2+5-\left(\frac{t_2}{2}+\frac{t_4}{4}+\frac{t_p}{p}+\frac{t_{p^2}}{p^2}+\frac{t_{2p}}{2p}+\frac{t_{2p^2}}{2p^2}\right)\right\}. \qquad \text{......(3.2.10)}$$

For the existence of a smooth epimorphism from a Fuchsian group Γ to M a necessary condition for $\gamma = 0$ is $t_4 \neq 0$. Since t_4 is always even, therefore for s = 5 we have,

(1) $t_4 = 4$ and other periods must be either p or p^2 as t_2, t_{2p}, t_{2p^2} do not take odd values individually.

(2) $t_4 = 2$ and the other periods must be i, j and k, where i, j, k can take values from the set $\{2, p, p^2, 2p, 2p^2\}$. In this case i, j and k may occur in the following ways:

(i) If i = j = k then say $t_i = 3$ and i is either p or p^2.

(ii) If $i \neq j$ and k = 0 then the possible periods of Γ are :

(a) i = 2, j = p [$t_2 = 2$, $t_p = 1$],

(b) i = 2, j = p^2 [$t_2 = 2$, $t_{p^2} = 1$],

(c) i = p, j = 2p [$t_p = 1$, $t_{2p} = 2$],

(d) i = p, j = $2p^2$ [$t_p = 1$, $t_{2p^2} = 2$],

(e) i = p, j = p^2 [$t_p = 1$, $t_{p^2} = 2$ or $t_p = 2$, $t_{p^2} = 1$],

(f) $i = p^2, j = 2p$ $[t_{p^2} = 1, t_{2p} = 2]$,

(g) $i = p^2, j = 2p^2$ $[t_{p^2} = 1, t_{2p^2} = 2]$

(iii) If $i \neq j \neq k$ then the possible periods of Γ are :

(a) $i = 2, j = p, k = 2p$,

(b) $i = 2, j = p, k = 2p^2$,

(c) $i = 2, j = p^2, k = 2p$,

(d) $i = 2, j = p^2, k = 2p^2$,

(e) $i = p, j = 2p, k = 2p^2$,

(f) $i = p^2, j = 2p, k = 2p^2$.

All the remaining periods are untenable according to theorem (3.1.1).

From the relation (3.2.10) we have

$$g-1 \geq 2p^2\left\{3-\left(\frac{t_4}{4}+\frac{t_i}{i}+\frac{t_j}{j}+\frac{t_k}{k}\right)\right\}. \qquad \text{......(3.2.11)}$$

If $t_4 = 4$ then the right hand side of (3.2.11) will be minimum when $i = p$ i.e., $t_p = 1$.

So from the relation (3.2.11) we get,

$$g-1 \geq 2p^2\left\{2-\frac{1}{p}\right\}$$

$$= 2p\{2p - 1\}$$

Therefore $g \geq 4p^2 - 2p + 1$.

If $t_4 = 2$ then the right hand side of (3.2.11) will be minimum only when $i = p(j = k = 0)$ i.e. $t_p = 3$.

Therefore from the relation (3.2.11) we get

$$g \geq 2p^2\left\{3-\frac{1}{2}-\frac{3}{p}\right\}+1$$

$$= p\{6p - p - 6\} + 1$$

$$= 5p^2 - 6p + 1.$$

Hence the minimum value of g for $s = 5$ is $5p^2 - 6p + 1$, which is also greater than $p^2 - 2p + 1$. Therefore g cannot have a lower minimum for $s = 5$ also. Therefore the minimum value for a compact Riemann surface admitting M as an automorphism group is $p^2 - 2p + 1$ and in that case M is a smooth quotient of a Fuchsian group Γ having the signature (4, 4, p).

We now summarize what we have proved in the form of a theorem.

Theorem 3.2.1 :

Let M be a Zs-metacyclic group of order $4p^2$ where p is an odd prime, with presentation :

$$\left\langle a, b : a^{p^2} = b^4 = 1, b^{-1}ab = a^{-1} \right\rangle.$$

Let M act as an automorphism group of some compact Riemann surface of genus $g \geq 2$. Then the minimum value of g and the corresponding signature of the Fuchsian group Γ of which M is a smooth quotient is :

$$g = p^2 - 2p + 1;\ (4, 4, p).$$

* * *

Chapter 4 : Finite smooth quotient of quadruple groups and some quadruple groups as automorphism groups of compact Riemann surfaces

CHAPTER - 4

FINITE SMOOTH QUOTIENT OF QUADRUPLE GROUPS AND SOME QUADRUPLE GROUPS AS AUTOMORPHISM GROUPS OF COMPACT RIEMANN SURFACES

Let us recall that in chapter 2 we defined a Fuchsian group with zero genus and three periods to be a Fuchsian triangle group. Similarly we can define a Fuchsian group with zero genus and four periods to be a Fuchsian quadruple group. So a quadruple Fuchsian group (m_1, m_2, m_3, m_4) is generated by four elements x_1, x_2, x_3, x_4 satisfying :

$$x_1^{m_1} = x_2^{m_2} = x_3^{m_3} = x_4^{m_4} = x_1x_2x_3x_4 = 1 \qquad ......(4.1)$$

or $$x_1^{m_1} = x_2^{m_2} = x_3^{m_3} = (x_1x_2x_3)^{m_4} = 1,$$

where m_1, m_2, m_3, m_4 are positive integers greater than 1 such that

$$\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} + \frac{1}{m_4} < 2$$

i.e., a quadruple Fuchsian group can be generated by three elements. Now smooth quotients of quadruple Fuchsian groups are also generated by three elements x, y, z satisfying :

$$x^{\ell} = y^m = z^n = (xyz)^{\xi} = 1$$

such that $\frac{1}{\ell} + \frac{1}{m} + \frac{1}{n} + \frac{1}{\xi} < 2$.

We call the smooth quotients of quadruple Fuchsian group an admissible quadruple group and ℓ, m, n, ξ will be called the periods of the group.

We have already mentioned in chapter 2 that a Fuchsian triangle group having periods which are relatively prime to each other can not have solvable smooth quotients. Similarly a Fuchsian quadruple groups with relatively prime periods cannot have solvable smooth quotients . So we will look for admissible quadruple groups with at least two periods not prime to each other, which are non-perfect quadruple groups.

In this chapter our aim is to find finite solvable admissible quadruple groups and in the last section of this chapter we find the solvable extension of admissible quadruple groups and corresponding genus of the surface to which these groups act as a group of automorphisms of compact Riemann surfaces.

In section 2.1 of chapter 2 we mentioned that a finite smooth quotient of any Fuchsian group occur as an automorphism group of a compact Riemann surface of genus ≥ 2. Therefore finite smooth quotients of (m_1, m_2, m_3, m_4) also occur as automorphism groups of compact Riemann surfaces. Here we consider some interesting class of admissible quadruple groups like (ℓ, m, n, μ) imposing some conditions among the integers ℓ, m, n, μ and construct infinitely many finite solvable quotients of the Fuchsian quadruple groups (ℓ, m, n, μ). Our results obtained in this chapter supersede those of Parbin's result [9] in the sense that those results come out as special cases of those of ours. Our technique seems to be applicable in case we take a Fuchsian group of genus zero and any number of periods thus ensuring the possibility of proving the existence of infinitely many finite solvable quotients of n-generator Fuchsian groups.

Here in this chapter also we need to find the signature of some normal subgroups of finite index of Fuchsian groups and the technique we use to find the signature is given in Lemma 2.1.1 of chapter 2. We also use the other Lemmas : Lemma 2.1.2, Lemma 2.1.3 and Lemma 2.1.4 included in chapter 2.

We now proceed to find an infinite family of solvable finite smooth quotients of Fuchsian quadruple groups (ℓ, m, n, μ) for the following classes of admissible quadruple groups :

Class I. (ℓ, m, n, μ) where $(\ell, m) = d_1 > 1$, $(\ell, n) = (\ell, \mu) = (m, n)$
$= (m, \mu) = (n, \mu) = 1$.

Class II. (ℓ, m, n, μ) where $(\ell, m) = d_1 > 1$, $(\ell, n) = d_2 > 1$ and
$(\ell, \mu) = (m, n) = (m, \mu) = (n, \mu) = 1$.

Class III. (ℓ, m, n, μ) where $(\ell, m) = d_1 > 1$, $(\ell, n) = d_2 > 1$,
$(\ell, \mu) = d_3 > 1$ and $(m, n) = (m, \mu) = (n, \mu) = 1$.

Class IV. (ℓ, m, n, μ) where $(\ell, m) = rd_1$, $(\ell, n) = rd_2$, $(m, n) = rd_3$, $(\ell, m, n) = r \geq 1$ and $(\ell, \mu) = (m, \mu) = (n, \mu) = 1$.

Class V. (ℓ, m, n, μ) where $(\ell, m) = d_1 > 1$, $(\ell, n) = d_2 > 1$, $(\ell, \mu) = d_3 > 1$, $(m, n) = d_4$ and $(m, \mu) = (n, \mu) = 1$.

The periods ℓ, m, n, μ are taken as unordered.

In the next section we find the above mentioned admissible quadruples occurring as smooth quotients of some quadruple Fuchsian groups and also we find the order of these admissible quadruple groups and the genus of the surface to which they act as a group of automorphisms of compact Riemann surfaces.

4.1 *Existence of smooth quotients of some families of quadruple groups*

This section is devoted in proving the existence of finite solvable smooth quotients of quadruple Fuchsian groups (ℓ, m, n, μ).

We now prove the theorems on the existence of solvable finite smooth quotients of Fuchsian quadruple groups.

Theorem 4.1.1 : Let $\Gamma = (\ell, m, n, \mu)$ be a Fuchsian group where ℓ, m, n, μ are positive integers greater than or equal to 2 such that $(\ell, m) = d_1 > 1$, $(\ell, n) = (\ell, \mu) = (m, n) = (m, \mu) = (n, \mu) = 1$. Then Γ admits a solvable smooth quotient of derived length 3 and of order $k^{2\gamma''} d_1 (n\mu)^{d_1-1} (ab)^{(n\mu)^{d_1-1}-1+2\gamma''}$ and genus

$\frac{1}{2} k^{2\gamma''} (ab)^{(n\mu)^{d_1-1}-2+2\gamma''} \left[2ab(\gamma''-1) + (n\mu)^{d_1-1}(2ab-b-a) \right] + 1$, where k is an integer ≥ 1 and

$$\gamma'' = \frac{n^{d_1-1}\mu^{d_1-1}}{2} \left[2d_1 - 2 - \frac{d_1}{n} - \frac{d_1}{\mu} \right] + 1.$$

Proof : Let $\ell = ad_1$, $m = bd_1$ where $(a, b) = 1$, $a \geq 1$, $b \geq 1$.

Let Γ be generated by elements x_1, x_2, x_3 and x_4 satisfying :

$$x_1^{ad_1} = x_2^{bd_1} = x_3^{n} = x_4^{\mu} = x_1 x_2 x_3 x_4 = 1.$$

Let u_1, u_2, u_3, u_4 be the images of x_1, x_2, x_3 and x_4 under the abelianizing homomorphism from Γ to Γ/Γ'. Then Γ/Γ' is generated by u_1, u_2, u_3 and u_4 satisfying the conditions :

$$u_1^{ad_1} = u_2^{bd_1} = u_3^n = u_4^\mu = u_1u_2u_3u_4 = 1$$

or equivalently,

$$u_1^{ad_1} = u_2^{bd_1} = u_3^n = (u_1u_2u_3)^\mu = 1.$$

Now $(u_1u_2u_3)^\mu = 1$ gives $u_1 = (u_2u_3)^{-1}$.

Therefore $\qquad u_1^{ad_1} = u_2^{ad_1}u_3^{ad_1} = u_2^{d_1}u_3 = 1$

$$\Rightarrow u_3 = u_2^{-d_1}.$$

Now $\qquad u_2^{bd_1} = u_3^n = \left(u_2^{-d_1}\right)^n = u_2^{-d_1n} = 1$

which gives $u_2^{d_1} = 1$.

Therefore $\Gamma/\Gamma' \cong Z_{d_1}$.

By Lemma (2.1.1) we have $\Gamma' = \left(\gamma'; a, b, \underbrace{n, \ldots, n}_{d_1\text{-times}}, \underbrace{\mu, \ldots, \mu}_{d_1\text{-times}}\right)$ where γ' is given by (2.1.1)

and we get $\gamma' = 0$.

So $$\Gamma' = \left(a, b, \underbrace{n, \ldots, n}_{d_1\text{-times}}, \underbrace{\mu, \ldots, \mu}_{d_1\text{-times}}\right). \qquad \text{......(4.1.1.1)}$$

Next let Γ' be generated by $y_1, y_2, y'_1, \ldots, y'_{d_1}, y''_1, \ldots, y''_{d_1}$ satisfying :

$$y_1^a = y_2^b = y_1'^n = \ldots = y_{d_1}'^n = y_1''^\mu = \ldots = y_{d_1}''^\mu = y_1y_2y'_1 \ldots y'_{d_1}y''_1 \ldots y''_{d_1} = 1.$$

Then $\Gamma'/\Gamma'' \cong \underbrace{Z_n \oplus \ldots \oplus Z_n}_{(d_1-1)\text{ summands}} \oplus \underbrace{Z_\mu \oplus \ldots \oplus Z_\mu}_{(d_1-1)\text{ summands}}$.

Application of Lemma (2.1.1) gives

$$\Gamma'' = \left(\gamma''; \underbrace{a, \ldots\ldots\ldots, a}_{(n\mu)^{d_1-1}-\text{times}}, \underbrace{b, \ldots\ldots\ldots, b}_{(n\mu)^{d_1-1}-\text{times}} \right) \qquad \ldots\ldots(4.1.1.2)$$

where γ'' is calculated from (2.1.1) and is

$$\gamma'' = \frac{n^{d_1-1}\mu^{d_1-1}}{2}\left[2d_1 - 2 - \frac{d_1}{n} - \frac{d_1}{\mu}\right] + 1. \qquad \ldots\ldots(4.1.1.3)$$

Therefore Γ'' satisfy the l.c.m. condition. The l.c.m. of the periods being ab = t, say. Hence by Lemma (2.1.3) Γ'' has a normal surface subgroup $\Gamma_s = \Gamma_{kt}\Gamma''$ of finite index for every positive integer $k \geq 1$.

Now $[\Gamma'' : \Gamma_s] = (kt)^{2\gamma''} a^{(n\mu)^{d_1-1}-1} b^{(n\mu)^{d_1-1}-1}, k \geq 1$.

The genus of Γ_s calculated from (2.1.1) is

$$\gamma_s = \frac{k^{2\gamma''}(ab)^{(n\mu)^{d_1-1}-2+2\gamma''}}{2}\left[2ab(\gamma''-1) + (n\mu)^{d_1-1}(2ab-b-a)\right] + 1, k \geq 1$$

We now set $G_s = \Gamma/\Gamma_s$ which is a smooth quotient of Γ and $G_s \trianglerighteq G_s' \trianglerighteq G_s'' \trianglerighteq G_s''' = \{1\}$.

Therefore G_s is a solvable group of length 3 and Γ admits a smooth quotient of order

$$|G_s| = \left|\Gamma/\Gamma_s\right| = \left|\Gamma/\Gamma'\right| \left|\Gamma'/\Gamma''\right| \left|\Gamma''/\Gamma_s\right|$$

$$= d_1(n\mu)^{d_1-1}(kt)^{2\gamma''}(ab)^{(n\mu)^{d_1-1}-1}$$

$$= k^{2\gamma''}d_1(n\mu)^{d_1-1}(ab)^{(n\mu)^{d_1-1}-1+2\gamma''} \qquad \ldots\ldots(4.1.1.4)$$

and of genus

$$\gamma_s = \frac{k^{2\gamma''}(ab)^{(n\mu)^{d_1-1}-2+2\gamma''}}{2}\left[2ab(\gamma''-1)+(n\mu)^{d_1-1}(2ab-b-a)\right]+1, k \geq 1$$

.......(4.1.1.5)

where $$\gamma'' = \frac{n^{d_1-1}\mu^{d_1-1}}{2}\left[2d_1 - 2 - \frac{d_1}{n} - \frac{d_1}{\mu}\right]+1 \neq 0.$$

This completes the proof.

The above theorem gives the following corollary.

Corollary 1 : When a = b = 1 then Γ'' is a surface subgroup of Γ and by Lemma (2.1.1) we get a family of solvable smooth quotient of Γ of derived length 3. In this case the order of the group is $k^{2\gamma''}d_1 n^{d_1-1}\mu^{d_1-1}$.

Theorem 4.1.2 : Let $\Gamma = (\ell, m, n, \mu)$ be a Fuchsian group where ℓ, m, n, μ are positive integers greater than or equal to 2 such that, $(\ell, m) = d_1 > 1$, $(\ell, n) = d_2 > 1$ and $(\ell, \mu) = (m, n) \neq (m, \mu) = (n, \mu) = 1$; d_1, d_2 are prime to each other. Then Γ admits a finite smooth quotients of order $d_1 d_2 AB$ where,

$$A = h_1^{d_2(d_1+1)-2} h_2^{d_1(d_2+1)-2} k^{2\gamma'}; k, h_1, h_2 \geq 1$$

and $$B = a^{A-1}\left(\frac{b}{h_1}\right)^{A/h_1 - 1}\left(\frac{c}{h_2}\right)^{A/h_2 - 1}\left(\frac{\mu}{h_3}\right)^{A/h_3 - 1}(t\ell')^{2\gamma_k}, \text{ where } \ell' = \frac{abc\mu}{h_1 h_2 h_3}$$

and of genus,

$$\frac{B}{2}\left[2\gamma_k - 2 + A\left(1 + \frac{1}{h_1} + \frac{1}{h_2} + \frac{1}{h_3} - \frac{1}{a} - \frac{1}{b} - \frac{1}{c} - \frac{1}{\mu}\right)\right]+1.$$

Proof : Let us take $\ell = ad_1d_2$, $m = bd_1$, $n = cd_2$ where a, b, c are prime to each other. Let Γ be generated by elements x_1, x_2, x_3 and x_4 satisfying :

$$x_1^{ad_1d_2} = x_2^{bd_1} = x_3^{cd_2} = x_4^{\mu} = x_1x_2x_3x_4 = 1.$$

Let u_1, u_2, u_3, u_4 be the images of x_1, x_2, x_3 and x_4 under the abelianizing homomorphism from Γ to Γ/Γ'. Then Γ/Γ' is generated by u_1, u_2, u_3 and u_4 satisfying the condition :

$$u_1^{ad_1d_2} = u_2^{bd_1} = u_3^{cd_2} = u_4^{\mu} = u_1u_2u_3u_4 = 1$$

or equivalently,

$$u_1^{ad_1d_2} = u_2^{bd_1} = u_3^{cd_2} = (u_1u_2u_3)^{\mu} = 1$$

Now $(u_1u_2u_3)^{\mu} = 1$ gives $u_1 = (u_2u_3)^{-1}$

$$u_1^{ad_1d_2} = (u_2u_3)^{ad_1d_2} = 1$$

$$\Rightarrow u_2^{d_1}u_3^{d_2} = 1$$

$$\Rightarrow u_2^{d_1} = u_3^{-d_2}.$$

Therefore,

$$u_2^{bd_1} = u_3^{cd_2} = (u_3^{d_2})^{c} = u_2^{d_1c} = 1$$

$$\Rightarrow u_2^{d_1} = 1.$$

Also $u_3^{d_2} = 1$.

Therefore $\Gamma/\Gamma' \cong Z_{d_1} \oplus Z_{d_2}$.

By Lemma (2.1.1) we have

$$\Gamma' = \left(\gamma'; a, \underbrace{b, b, \ldots, b}_{d_2-\text{times}}, \underbrace{c, \ldots, c}_{d_1-\text{times}}, \underbrace{\mu, \ldots, \mu}_{d_1d_2-\text{times}} \right) \quad \text{......(4.1.2.1)}$$

where γ' is given by (2.1.1) and we get

$$\gamma' = \frac{1}{2}(d_1 - 1)(d_2 - 1) > 0 \quad (\text{since } d_1 > 1 \text{ and } d_2 > 1).$$

By Lemma (2.1.4), Γ' has a subgroup Γ_k, $k > 1$ of finite index whose periods satisfy the l.c.m. condition.

Let $\Gamma_k^* = \{x^k, x \in \Gamma'\}$ be a subgroup of Γ' for $k \geq 2$. Let $\Gamma_k = \Gamma_k^*\Gamma''$, then by Lemma

(2.1.3) Γ_k' is normal in Γ' of finite index.

As $\Gamma'' \subseteq \Gamma_k \subseteq \Gamma'$, so ${}^{\Gamma'}\!/\!{}_{\Gamma_k}$ is abelian.

Consider an abelianizing homomorphism

$$\phi : \Gamma' \to {}^{\Gamma'}\!/\!{}_{\Gamma_k}.$$

Let $u,, u_1', \ldots, u_{d_2}', v_1', \ldots, v_{d_1}', w_1', \ldots, w_{d_1d_2}', a_1, b_1, \ldots, a_{\gamma'}, b_{\gamma'}$ be the images of $x, x_1', \ldots, x_{d_2}', y_1', \ldots, y_{d_1}', z_1', \ldots, z_{d_1d_2}', \alpha_1, \beta_1, \ldots, \alpha_{\gamma'}, \beta_{\gamma'}$ respectively under the above mentioned homomorphism satisfying the conditions :

$$u^a = u_1'^b = \ldots = u_{d_2}'^b = v_1'^c = \ldots = v_{d_1}'^c = w_1'^\mu = \ldots = w_{d_1d_2}'^\mu = uu_1' \ldots u_{d_2}' v_1' \ldots v_{d_1}' w_1' \ldots w_{d_1d_2}' = 1$$

and $u^k = u_1'^k = \ldots = u_{d_2}'^k = v_1'^k = \ldots = v_{d_1}'^k = w_1'^k = \ldots = w_{d_1d_2}'^k = a_1^k = b_1^k = \ldots = a_{\gamma'}^k = b_{\gamma'}^k = 1$

where the elements commute with each other.

If $(b, k) = h_1 \geq 1$, $(c, k) = h_2 \geq 1$ and $(\mu, k) = h_3 \geq 1$ then the above relation gives :

$$u_1'^{h_1} = \ldots = u_{d_2}'^{h_1} = v_1'^{h_2} = \ldots = v_{d_1}'^{h_2} = w_1'^{h_3} = \ldots = w_{d_1d_2}'^{h_3} = 1$$

and the elements commute with each other.

Therefore $${}^{\Gamma'}\!/\!{}_{\Gamma_k} \cong \underbrace{z_{h_1} \oplus \ldots \oplus z_{h_1}}_{(d_2-1)\text{ summands}} \oplus \underbrace{z_{h_2} \oplus \ldots \oplus z_{h_2}}_{(d_1-1)\text{ summands}} \oplus \underbrace{z_{h_3} \oplus \ldots \oplus z_{h_3}}_{(d_1d_2-1)\text{ summands}} \oplus \underbrace{z_k \oplus \ldots \oplus z_k}_{2\gamma'\text{ summands}}.$$

Therefore, $$\left| {}^{\Gamma'}\!/\!{}_{\Gamma_k} \right| = h_1^{d_2-1} h_2^{d_1-1} (h_1h_2)^{d_1d_2-1} k^{2\gamma'}$$

$$= h_1^{d_2+d_1d_2-2} h_2^{d_1+d_1d_2-2} k^{2\gamma'}$$

$$= h_1^{d_2(1+d_1)-2} h_2^{d_1(1+d_2)-2} k^{2\gamma'}$$

$$= A, \text{ say.} \qquad \ldots\ldots(4.1.2.2)$$

By application of Lemma (2.1.1) we get,

$$\Gamma_k = \left(\gamma_k ; \underbrace{a,\ldots,a}_{A\text{-times}}, \underbrace{\frac{b}{h_1},\ldots,\frac{b}{h_1}}_{A/h_1\text{-times}}, \underbrace{\frac{c}{h_2},\ldots,\frac{c}{h_2}}_{A/h_2\text{-times}}, \underbrace{\frac{\mu}{h_3},\ldots,\frac{\mu}{h_3}}_{A/h_3\text{-times}} \right) \qquad \ldots\ldots(4.1.2.3)$$

The periods of Γ_k satisfy the l.c.m. condition, therefore Γ_k' is a surface group. The genus of γ_k is calculated from (2.1.1) and we have

$$\gamma_k = \frac{A}{2}\left[2\gamma' - 2 + d_1\left(1-\frac{1}{c}\right) + d_2\left(1-\frac{1}{b}\right) + d_1 d_2\left(1-\frac{1}{\mu}\right) \right.$$

$$\left. - \frac{1}{h_1} - \frac{1}{h_2} - \frac{1}{h_3} + \frac{1}{b} + \frac{1}{c} + \frac{1}{\mu} \right] + 1.$$

By Lemma (2.1.3), let us construct another subgroup Γ_k^* generated by $t^{th}(t > 1)$ power of the infinite order generators of Γ_k.

Let $N = \Gamma_k^* \Gamma_k'$, then N is normal in Γ_k of finite index and Γ_k / N is abelian as

$\Gamma_k' \subseteq \Gamma_k^* \Gamma_k' = N.$

Now we have,

$$\Gamma_k / N \cong \underbrace{Z_a \oplus \ldots \oplus Z_a}_{(A-1)\text{ summands}} \oplus \underbrace{Z_{b/h_1} \oplus \ldots \oplus Z_{b/h_1}}_{\left(A/h_1 - 1\right)\text{summands}} \oplus \underbrace{Z_{c/h_2} \oplus \ldots \oplus Z_{c/h_2}}_{\left(A/h_2 - 1\right)\text{summands}}$$

$$\oplus \underbrace{Z_{\mu/h_3} \oplus \ldots \oplus Z_{\mu/h_3}}_{\left(A/h_3 - 1\right)\text{summands}} \oplus \underbrace{Z_{t\ell'} \oplus \ldots \oplus Z_{t\ell'}}_{2\gamma_k\text{ summands}}$$

where $\ell' = \dfrac{abc\mu}{h_1 h_2 h_3}$ is the l.c.m. of the periods of Γ_k.

Therefore $$\left| \Gamma_k / N \right| = a^{A-1} \left(\frac{b}{h_1}\right)^{A/h_1 - 1} \left(\frac{c}{h_2}\right)^{A/h_2 - 1} \left(\frac{\mu}{h_3}\right)^{A/h_3 - 1} (t\ell')^{2\gamma_k}$$

$$= B, \text{ say.} \qquad \ldots\ldots(4.1.2.4)$$

By application of Lemma (2.1.1) we have

$$N = (\gamma_n; \ldots).$$

Since N is a surface group and therefore $G = \Gamma/N$ is a smooth quotient of Γ and the genus is obtained from (2.1.1) is

$$\gamma_n = \frac{B}{2}\left[2\gamma_k - 2 + A\left(1 + \frac{1}{h_1} + \frac{1}{h_2} + \frac{1}{h_3} - \frac{1}{a} - \frac{1}{b} - \frac{1}{c} - \frac{1}{\mu}\right)\right] + 1$$

and the order of G is

$$|G| = \left|\Gamma/N\right| = \left|\Gamma/\Gamma'\right|\left|\Gamma'/\Gamma_k\right|\left|\Gamma_k/N\right|$$

$$= d_1 d_2 AB$$

where A and B are given by (4.1.2.2) and (4.1.2.4) respectively. Thus Γ admits smooth quotients, but we cannot say that whether it is solvable or not.

This completes the proof of the theorem.

The above theorem arises the following corollaries :

Corollary 1 : If a = b = c = 1 in theorem 4.1.2, then Γ admits a metabelian smooth quotients of order $d_1 d_2 \mu^{d_1 d_2 + 2\gamma' - 1} k^{2\gamma'}, k \geq 1$ and of genus

$$\frac{1}{2}\mu^{d_1 d_2 + 2\gamma' - 1} k^{2\gamma'}\left[2\gamma' - 2 + d_1 d_2\left(1 - \frac{1}{\mu}\right)\right] + 1, k \geq 1$$

where $\gamma' = \frac{1}{2}(d_1 - 1)(d_2 - 1)$.

Proof : From (4.1.2.1) we have $\Gamma' = \left(\gamma; \underbrace{\mu, \ldots, \mu}_{d_1 d_2 - \text{times}}\right)$ which shows that Γ'' is a surface group, since Γ' satisfies the l.c.m. condition.

By applying Lemma (2.1.3), let us define a subgroup

$\Gamma_k = \{x^{\mu k},\ x \in \Gamma'$ for any positive integer $k \geq 1\}$ of Γ',

where μ is the l.c.m. of the periods of Γ'.

Let $N = \Gamma_k\Gamma''$ and N is normal in Γ'.

Further $\Gamma'' \subseteq \Gamma_k\Gamma'' = N$.

Therefore Γ'/N is abelian as it contains Γ''. Considering an abelianizing homomorphism

$$\phi : \Gamma' \to \Gamma'/N$$

we have $$\Gamma'/N \cong \underbrace{z_\mu \oplus \ldots \oplus z_\mu}_{(d_1d_2-1)\text{ summands}} \oplus \underbrace{z_{\mu k} \oplus \ldots \oplus z_{\mu k}}_{2\gamma'\text{ summands}}$$

Therefore $\left|\Gamma'/N\right| = \mu^{d_1d_2-1}(\mu k)^{2\gamma'} = \mu^{d_1d_2+2\gamma'-1}k^{2\gamma'}$.

Applying Lemma (2.1.1) we get $N = (\gamma''; \ldots)$, which shows that N contains no finite order generators. Thus N is a surface group and the genus γ'' is calculated from (2.1.1) is

$$\frac{1}{2}\mu^{d_1d_2+2\gamma'-1}k^{2\gamma'}\left[2\gamma' - 2 + d_1d_2\left(1 - \frac{1}{\mu}\right)\right] + 1 \quad \text{for } k \geq 1$$

where $\gamma' = \frac{1}{2}(d_1 - 1)(d_2 - 1)$.

If we set $G = \Gamma/N$, then G is smooth quotient of Γ as N is a normal surface group.

Also $G' = \Gamma'/N$, $G'' = \Gamma''/N = \{1\}$ as $\Gamma'' \subseteq N$.

Therefore $G \trianglerighteq G' \trianglerighteq G'' = \{1\}$ showing that G is metabelian.

Now $|G| = \left|\Gamma/\Gamma'\right|\left|\Gamma'/N\right| = d_1d_2\mu^{d_1d_2+2\gamma'-1}k^{2\gamma'}, k \geq 1$.

Corollary 2 : **Let a = 1 and b, c > 1.**

From (4.1.2.1) we have $\Gamma' = \left(\gamma'; \underbrace{b,\ldots,b}_{d_2-\text{times}}, \underbrace{c,\ldots,c}_{d_1-\text{times}}, \underbrace{\mu,\ldots,\mu}_{d_1d_2-\text{times}} \right)$.

Therefore Γ'' is a surface group as the periods of Γ' satisfies the l.c.m. condition.

By applying Lemma (2.1.3), let us define a subgroup

$\Gamma_k = \{x^{bc\mu k},\ x \in \Gamma'$ for any positive integer $k \geq 1\}$ of Γ',

where $bc\mu k$ is the l.c.m. of the periods of Γ'.

Let $N = \Gamma_k\Gamma''$ and N is normal in Γ'.

Further $\Gamma'' \subseteq \Gamma_k\Gamma'' = N$.

Therefore Γ'/N is abelian as it contains Γ''.

Considering an abelianizing homomorphism $\phi : \Gamma' \rightarrow \Gamma'/N$ we have

$$\Gamma'/N \cong \underbrace{Z_b \oplus \ldots \oplus Z_b}_{(d_2-1)\text{ summands}} \oplus \underbrace{Z_c \oplus \ldots \oplus Z_c}_{(d_1-1)\text{ summands}} \oplus \underbrace{Z_\mu \oplus \ldots \oplus Z_\mu}_{(d_1d_2-1)\text{ summands}} \oplus \underbrace{Z_{bc\mu k} \oplus \ldots \oplus Z_{bc\mu k}}_{2\gamma'\text{ summands}}.$$

Therefore,

$$\left|\Gamma'/N\right| = b^{d_2-1}c^{d_1-1}\mu^{d_1d_2-1}(bc\mu k)^{2\gamma'}$$

$$= b^{d_2+2\gamma'-1}c^{d_1+2\gamma'-1}\mu^{d_1d_2+2\gamma'-1}k^{2\gamma'}.$$

Applying Lemma (2.1.1) we get $N = (\gamma''; \ldots)$ which shows that N contains no finite order generators. Thus N is a surface group and the genus γ'' is calculated from (2.1.1) is

$$\frac{1}{2}b^{d_2+2\gamma'-1}c^{d_1+2\gamma'-1}\mu^{d_1d_2+2\gamma'-1}k^{2\gamma'}\left[2\gamma' - 2 + d_1 + d_2 + d_1d_2 - \frac{d_1}{c} - \frac{d_2}{b} - \frac{d_1d_2}{\mu}\right] + 1,\ k \geq 1$$

where $\gamma' = \frac{1}{2}(d_1 - 1)(d_2 - 1)$.

If we set $G = \Gamma/N$, then G is a smooth quotient of Γ and N is a normal surface group.

Also $G' = \Gamma'/N, G'' = \Gamma''/N = \{1\}$ as $\Gamma'' \subseteq N$.

Therefore $G \trianglerighteq G' \trianglerighteq G'' = \{1\}$ showing that G is metabelian.

Now $|G| = \left|\Gamma/\Gamma'\right| \left|\Gamma'/N\right| = d_1 d_2 b^{d_2+2\gamma'-1} c^{d_1+2\gamma'-1} \mu^{d_1 d_2+2\gamma'-1} k^{2\gamma'}$.

Theorem 4.1.3 : Let $\Gamma = (\ell, m, n, \mu)$ be a Fuchsian group where ℓ, m, n, μ are positive integers greater than or equal to 2 such that

$$(\ell, m) = d_1 > 1, \; (\ell, n) = d_2 > 1, \; (\ell, \mu) = d_3 > 1$$

and $\quad (m, n) = (m, \mu) = (n, \mu) = 1,$

where d_1, d_2, d_3 are prime to each other. Then Γ admits a finite smooth quotient of Γ of genus

$$\frac{B}{2}\left[2\gamma_k - 2 + A\left(1 + \frac{1}{h_1} + \frac{1}{h_2} + \frac{1}{h_3} - \frac{1}{a} - \frac{1}{b} - \frac{1}{c} - \frac{1}{r}\right)\right] + 1$$

and the order is,

$$d_1 d_2 d_3 AB$$

where $\quad A = h_1^{d_2 d_3 - 1} h_2^{d_1 d_3 - 1} h_3^{d_1 d_2 - 1} k^{2\gamma'}$

$$B = a^{A-1} \left(\frac{b}{h_1}\right)^{A/h_1 - 1} \left(\frac{c}{h_2}\right)^{A/h_2 - 1} \left(\frac{r}{h_3}\right)^{A/h_3 - 1} (t\ell')^{2\gamma_k}$$

$$\ell' = \frac{abcr}{h_1 h_2 h_3}$$

and

$$\gamma_k = \frac{A}{2}\left[2\gamma' - 2 + d_2 d_3\left(1 - \frac{1}{b}\right) + d_1 d_3\left(1 - \frac{1}{c}\right) + d_1 d_2\left(1 - \frac{1}{r}\right) - \frac{1}{h_1} - \frac{1}{h_2} - \frac{1}{h_3} + \frac{1}{b} + \frac{1}{c} + \frac{1}{r}\right] + 1.$$

Proof : Let us take $\ell = ad_1d_2d_3$, $m = bd_1$, $n = cd_2$, $\mu = rd_3$ where a, b, c, r are pairwise rime to each other.

Let Γ be generated by elements x_1, x_2, x_3 and x_4 satisfying :

$$x_1^{ad_1d_2d_3} = x_2^{bd_1} = x_3^{cd_2} = x_4^{rd_3} = x_1x_2x_3x_4 = 1$$

or equivalently,

$$x_1^{ad_1d_2d_3} = x_2^{bd_1} = x_3^{cd_2} = (x_1x_2x_3)^{rd_3} = 1.$$

Let u_1, u_2, u_3 and u_4 be the images of x_1, x_2, x_3 and x_4 respectively under the abelianizing homomorphism from Γ onto Γ/Γ'. Then Γ/Γ' is generated by u_1, u_2, u_3 and u_4 satisfying :

$$u_1^{ad_1d_2d_3} = u_2^{bd_1} = u_3^{cd_2} = u_4^{rd_3} = u_1u_2u_3u_4 = 1$$

or equivalently,

$$u_1^{ad_1d_2d_3} = u_2^{bd_1} = u_3^{cd_2} = (u_1u_2u_3)^{rd_3} = 1.$$

We have $u_2^{d_1} = 1, u_3^{d_2} = 1, u_4^{d_3} = 1$.

Therefore $\Gamma/\Gamma' \cong z_{d_1} \oplus z_{d_2} \oplus z_{d_3}$.

By Lemma (2.1.1) we have,

$$\Gamma' = \left(\gamma'; a, \underbrace{b,\ldots,b}_{d_2d_3-\text{times}}, \underbrace{c,\ldots,c}_{d_1d_3-\text{times}}, \underbrace{r,\ldots,r}_{d_1d_2-\text{times}} \right) \qquad ......(4.1.3.1)$$

where γ' is given by (2.1.1) and is

$$\gamma' = \frac{1}{2}[2d_1d_2d_3 - d_2d_3 - d_1d_3 - d_1d_2 + 1].$$

By Lemma (2.1.4), Γ' has a subgroup Γ_k, $k > 1$ of finite index whose periods satisfy the l.c.m. condition.

Let $\Gamma_k^* = \{x^k,\ x \in \Gamma'\}$ be a subgroup of Γ', for $k \geq 2$.

Let $\Gamma_k = \Gamma_k^*\Gamma''$, then by Lemma (2.1.3) Γ_k' is normal in Γ' of finite index.

As $\Gamma'' \subseteq \Gamma_k \subseteq \Gamma'$, so Γ'/Γ_k is abelian.

Consider an abelianizing homomorphism :

$$\phi : \Gamma' \to \Gamma'/\Gamma_k .$$

Let $u,, u_1',\ldots,u_{d_2d_3}', v_1',\ldots,v_{d_1d_3}', w_1',\ldots,w_{d_1d_2}', a_1, b_1,\ldots,a_{\gamma'}, b_{\gamma'}$ be the images of $x, x_1',\ldots,x_{d_2d_3}', y_1',\ldots,y_{d_1d_3}', z_1',\ldots,z_{d_1d_2}', \alpha_1, \beta_1,\ldots,\alpha_{\gamma'}, \beta_{\gamma'}$ respectively under the above mentioned homomorphism satisfying the conditions :

$$u^a = u_1'^b = \ldots = u_{d_2d_3}'^b = v_1'^c = \ldots = v_{d_1d_3}'^c = w_1'^r = \ldots = w_{d_1d_2}'^r$$

$$= uu_1'\ldots u_{d_2d_3}' v_1'\ldots v_{d_1d_3}' w_1'\ldots w_{d_1d_2}' = 1$$

and $$u^k = u_1'^k = \ldots = u_{d_2d_3}'^k = v_1'^k = \ldots = v_{d_1d_3}'^k = w_1'^k = \ldots = w_{d_1d_2}'^k =$$

$$a_1^k = b_1^k = \ldots = a_{\gamma'}^k = b_{\gamma'}^k = 1$$

where the elements commute with each other.

If $(b, k) = h_1 \geq 1$, $(c, k) = h_2 \geq 1$ and $(r, k) = h_3 \geq 1$ then the above relation gives :

$$u_1'^{h_1} = \ldots = u_{d_2d_3}'^{h_1} = v_1'^{h_2} = \ldots = v_{d_1d_3}'^{h_2} = w_1'^{h_3} = \ldots = w_{d_1d_2}'^{h_3} = 1$$

and the elements commute with each other.

Therefore,

$$\Gamma'/\Gamma_k \cong \underbrace{z_{h_1} \oplus \ldots \oplus z_{h_1}}_{(d_2d_3-1)\text{ summands}} \oplus \underbrace{z_{h_2} \oplus \ldots \oplus z_{h_2}}_{(d_1d_3-1)\text{ summands}} \oplus \underbrace{z_{h_3} \oplus \ldots \oplus z_{h_3}}_{(d_1d_2-1)\text{ summands}} \oplus \underbrace{z_k \oplus \ldots \oplus z_k}_{2\gamma'\text{ summands}} .$$

And $$\left|\Gamma'/\Gamma_k\right| = h_1^{d_2d_3-1} h_2^{d_1d_3-1} h_3^{d_1d_2-1} k^{2\gamma'}$$

$$= A, \text{ say.} \qquad \ldots\ldots(4.1.3.2)$$

By application of Lemma (2.1.1) we get,

$$\Gamma_k = \left(\gamma_k ; \underbrace{a,\ldots,a}_{A\text{-times}}, \underbrace{\frac{b}{h_1},\ldots,\frac{b}{h_1}}_{A/h_1\text{-times}}, \underbrace{\frac{c}{h_2},\ldots,\frac{c}{h_2}}_{A/h_2\text{-times}}, \underbrace{\frac{r}{h_3},\ldots,\frac{r}{h_3}}_{A/h_3\text{-times}}\right). \qquad \ldots\ldots(4.1.3.3)$$

The periods of Γ_k satisfy the l.c.m. condition, therefore Γ_k' is a surface group. The genus of Γ_k is calculated from (2.1.1) and we get

$$\gamma_k = \frac{A}{2}\left[2\gamma' - 2 + d_2d_3\left(1-\frac{1}{b}\right) + d_1d_3\left(1-\frac{1}{c}\right) + d_1d_2\left(1-\frac{1}{r}\right) - \frac{1}{h_1} - \frac{1}{h_2} - \frac{1}{h_3} + \frac{1}{b} + \frac{1}{c} + \frac{1}{r}\right] + 1$$

where A is given by (4.1.3.2).

Let us construct another subgroup (by applying Lemma (2.1.3)) Γ_k^* generated by t^{th} $(t > 1)$ power of the infinite order generators of Γ_k.

Let $N = \Gamma_k^*\Gamma_k'$, then N is normal in Γ_k of finite index and Γ_k/N is abelian as

$$\Gamma_k' \subseteq \Gamma_k^*\Gamma_k' = N.$$

Now we have,

$$\Gamma_k/N \cong \underbrace{Z_a \oplus \ldots \oplus Z_a}_{(A-1)\,\text{summands}} \oplus \underbrace{Z_{b/h_1} \oplus \ldots \oplus Z_{b/h_1}}_{\left(A/h_1 - 1\right)\text{summands}} \oplus \underbrace{Z_{c/h_2} \oplus \ldots \oplus Z_{c/h_2}}_{\left(A/h_2 - 1\right)\text{summands}} \oplus \underbrace{Z_{r/h_3} \oplus \ldots \oplus Z_{r/h_3}}_{\left(A/h_3 - 1\right)\text{summands}} \oplus \underbrace{Z_{t\ell'} \oplus \ldots \oplus Z_{t\ell'}}_{2\gamma_k\,\text{summands}}$$

where $\ell' = \dfrac{abcr}{h_1h_2h_3}$ is the l.c.m. of the periods of Γ_k.

Therefore $$\left|\Gamma_k/N\right| = a^{A-1}\left(\frac{b}{h_1}\right)^{A/h_1-1}\left(\frac{c}{h_2}\right)^{A/h_2-1}\left(\frac{r}{h_3}\right)^{A/h_3-1}(t\ell')^{2\gamma_k}$$

$$= B, \text{ say.} \qquad \text{......(4.1.3.4)}$$

By application of Lemma (2.1.1) we have

$N = (\gamma_n; \ldots)$

Since N is a surface group and therefore $G = \Gamma/N$ is a smooth quotient of Γ and the genus is obtained from (2.1.1) is

$$\gamma_n = \frac{B}{2}\left[2\gamma_k - 2 + A\left(1 + \frac{1}{h_1} + \frac{1}{h_2} + \frac{1}{h_3} - \frac{1}{a} - \frac{1}{b} - \frac{1}{c} - \frac{1}{r}\right)\right] + 1$$

where A and B are given by (4.1.3.2) and (4.1.3.4) respectively.

Now the order of G is,

$$|G| = \left|\Gamma/N\right| = \left|\Gamma/\Gamma'\right|\left|\Gamma'/\Gamma_k\right|\left|\Gamma_k/N\right|$$

$$= d_1 d_2 d_3 AB$$

where A and B are given by (4.1.3.2) and (4.1.3.4) respectively. Thus Γ admits smooth quotients, but we cannot say that whether it is solvable or not. This completes the proof of the theorem.

The above theorem arises the following corollaries :

Corollary 1 : If a = b = c = r = 1 in theorem 4.1.3, then Γ admits abelian smooth quotient of order $d_1 d_2 d_3$ and of genus

$$\frac{1}{2}[2d_1 d_2 d_3 - d_2 d_3 - d_1 d_3 - d_1 d_2 + 1],$$

as well as metabelian smooth quotients of order $d_1 d_2 d_3 k^{2\gamma'}$ and of genus

$$k^{2\gamma'}\left[\frac{1}{2}\{2d_1d_2d_3 - d_2d_3 - d_1d_3 - d_1d_2 - 1\}\right]+1, \text{ for } k \geq 1.$$

Proof : From (4.1.3.1) we have $\Gamma' = (\gamma'; \ldots)$ which shows that Γ admits abelian smooth quotient of order $d_1d_2d_3$ and of genus

$$\frac{1}{2}[2d_1d_2d_3 - d_2d_3 - d_1d_3 - d_1d_2 + 1].$$

Now we apply Lemma (2.1.2) and define a subgroup

$\Gamma_k = \{x^k, x \in \Gamma' \text{ for } k \geq 1\}$ of Γ' containing Γ''.

Let $N = \Gamma_k\Gamma''$, then N is normal subgroup of Γ of finite index.

As $\Gamma'' \subseteq \Gamma_k\Gamma'' = N$, therefore Γ/N is abelian.

Consider an abelianizing homomorphism :

$\phi : \Gamma' \to \Gamma'/N$ we have

$$\Gamma'/N \cong \underbrace{z_k \oplus \ldots \oplus z_k}_{2\gamma' \text{ summands}}.$$

Applying Lemma (2.1.1) we have $N = (\gamma''; \ldots)$ which shows that N contains no finite order generators. Thus N is a surface group. The genus γ'' is calculated from (2.1.1) is

$$\gamma'' = k^{2\gamma'}\left[\frac{1}{2}[2d_1d_2d_3 - d_2d_3 - d_1d_3 - d_1d_2 - 1]\right]+1, k \geq 1.$$

If we set $G = \Gamma/N$ then G is a smooth quotient of Γ, since N is a surface group.

Also $G' = \Gamma'/N, G'' = \Gamma''/N = \{1\}$ as $\Gamma'' \subseteq N$.

Therefore $G \trianglerighteq G' \trianglerighteq G'' = \{1\}$. Hence G is metabelian.

Now the order of G is

$$|G| = \left|{}^{\Gamma}\!/_{N}\right| = \left|{}^{\Gamma}\!/_{\Gamma'}\right|\left|{}^{\Gamma'}\!/_{N}\right|$$

$$= d_1 d_2 d_3 k^{2\gamma'}, \text{ for } k \geq 1$$

where $\gamma' = \frac{1}{2}[2d_1d_2d_3 - d_2d_3 - d_1d_3 - d_1d_2 + 1]$.

Corollary 2 : **Let a = 1 and b, c, r > 1.**

From (4.1.3.1) we have

$$\Gamma' = \left(\gamma'; \underbrace{b,\ldots,b}_{d_2d_3-\text{times}}, \underbrace{c,\ldots,c}_{d_1d_3-\text{times}}, \underbrace{r,\ldots,r}_{d_1d_2-\text{times}} \right).$$

Therefore Γ'' is a surface group as the periods of Γ' satisfies the l.c.m. condition.

By applying Lemma (2.1.3), let us define a subgroup

$\Gamma_k = \{x^{bcrk}, x \in \Gamma'$ for any positive integer $k \geq 1\}$ of Γ',

where bcr is the l.c.m. of the periods of Γ'.

Let $N = \Gamma_k\Gamma''$ and N is normal in Γ'.

Further $\Gamma'' \subseteq \Gamma_k\Gamma'' = N$.

Therefore ${}^{\Gamma'}\!/_{N}$ is abelian as it contains Γ''.

Considering an abelianizing homomorphism :

$$\phi : \Gamma' \to {}^{\Gamma'}\!/_{N}$$

we have $${}^{\Gamma'}\!/_{N} \cong \underbrace{z_b \oplus \ldots \oplus z_b}_{(d_2d_3-1)\text{ summands}} \oplus \underbrace{z_c \oplus \ldots \oplus z_c}_{(d_1d_3-1)\text{ summands}} \oplus \underbrace{z_r \oplus \ldots \oplus z_r}_{(d_1d_2-1)\text{ summands}} \oplus \underbrace{z_{bcrk} \oplus \ldots \oplus z_{bcrk}}_{2\gamma'\text{ summands}}$$

Therefore $$\left|{}^{\Gamma'}\!/_{N}\right| = b^{d_2d_3-1}c^{d_1d_3-1}r^{d_1d_2-1}(bcrk)^{2\gamma'}$$

$$= b^{d_2d_3+2\gamma'-1}c^{d_1d_3+2\gamma'-1}r^{d_1d_2+2\gamma'-1}k^{2\gamma'}, \ k \geq 1.$$

Applying Lemma (2.1.1) we get N = $(\gamma''; \ldots)$ which shows that N contains no finite order generators. Thus N is a surface group and the genus γ'' is calculated from (2.1.1) is

$$\frac{1}{2} b^{d_2 d_3 + 2\gamma' - 1} c^{d_1 d_3 + 2\gamma' - 1} r^{d_1 d_2 + 2\gamma' - 1} k^{2\gamma'} \left[2d_1 d_2 d_3 - \frac{d_2 d_3}{b} - \frac{d_1 d_3}{c} - \frac{d_1 d_2}{r} - 1 \right] + 1, \text{ for } k \geq 1,$$

where $\gamma' = \frac{1}{2}[2d_1 d_2 d_3 - d_2 d_3 - d_1 d_3 - d_1 d_2 + 1]$.

If we set $G = \Gamma/N$, then G is a smooth quotient of Γ as N is a normal surface group.

Also $G' = \Gamma'/N, G'' = \Gamma''/N = \{1\}$ as $\Gamma'' \subseteq N$.

Therefore $G \trianglerighteq G' \trianglerighteq G'' = \{1\}$ showing that G is metabelian.

Now $\quad |G| = \left|\Gamma/N\right|$.

$$= \left|\Gamma/\Gamma'\right| \left|\Gamma'/N\right|$$

$$= d_1 d_2 d_3 b^{d_2 d_3 + 2\gamma' - 1} c^{d_1 d_3 + 2\gamma' - 1} r^{d_1 d_2 + 2\gamma' - 1} k^{2\gamma'}, \text{ for } k \geq 1.$$

Theorem 4.1.4 : Let $\Gamma = (\ell, m, n, \mu)$ be a Fuchsian group where ℓ, m, n, μ are positive integers ≥ 2 such that

$$(\ell, m) = rd_1, (\ell, n) = rd_2, (m, n) = rd_3, (\ell, m, n) = r \geq 1$$

and $\quad (\ell, \mu) = (m, \mu) = (n, \mu) = 1$,

where d_1, d_2, d_3 are pairwise prime to each other. Then Γ admits a metabelian smooth quotient of order

$$Ar^2 d_1^2 d_2 d_3$$

and of genus

$$\frac{A}{2}\left[2r^2 d_1^2 d_2 d_3 - \frac{rd_1 d_3}{a} - \frac{rd_1 d_2}{b} - \frac{rd_1^2}{c} - \frac{r^2 d_1^2 d_2 d_3}{\mu} \right] + 1$$

where $$A = a^{rd_1d_3+2\gamma'-1}b^{rd_1d_2+2\gamma'-1}c^{rd_1^2+2\gamma'-1}\mu^{r^2d_1^2d_2d_3+2\gamma'-1}k^{2\gamma'}, k \geq 1$$

and $$\gamma' = \frac{1}{2}r[rd_1^2d_2d_3 - d_1d_3 - d_1d_2 - d_1^2]+1.$$

Proof : Let us assume that $\ell = ard_1d_2$, $m = brd_1d_3$, $n = crd_2d_3$ where a, b, c are pairwise prime to each other.

Let Γ be generated by elements x_1, x_2, x_3 and x_4 satisfying :

$$x_1^{\ell} = x_2^{m} = x_3^{n} = x_4^{\mu} = x_1x_2x_3x_4 = 1$$

i.e. $$x_1^{ard_1d_2} = x_2^{brd_1d_3} = x_3^{crd_2d_3} = x_4^{\mu} = x_1x_2x_3x_4 = 1$$

or equivalently,

$$x_1^{ard_1d_2} = x_2^{brd_1d_3} = x_3^{crd_2d_3} = (x_1x_2x_3)^{\mu} = 1.$$

Let u_1, u_2, u_3 and u_4 be the images of x_1, x_2, x_3 and x_4 respectively under the abelianizing homomorphism from Γ to Γ/Γ'. Then Γ/Γ' is generated by u_1, u_2, u_3 and u_4 satisfying :

$$u_1^{ard_1d_2} = u_2^{brd_1d_3} = u_3^{crd_2d_3} = u_4^{\mu} = u_1u_2u_3u_4 = 1$$

or equivalently,

$$u_1^{ard_1d_2} = u_2^{brd_1d_3} = u_3^{crd_2d_3} = (u_1u_2u_3)^{\mu} = 1.$$

Now $(u_1u_2u_3)^{\mu} = 1$ gives $u_3 = (u_1u_2)^{-1}$.

Therefore,

$$u_3^{crd_2d_3} = (u_1u_2)^{crd_2d_3} = 1 \Rightarrow u_1^{rd_2}u_2^{rd_3} = 1$$

i.e. $$u_1^{rd_2} = u_2^{-rd_3}.$$

Now $$u_1^{ard_1d_2} = \left(u_1^{rd_2}\right)^{ad_1} = 1 \Rightarrow u_2^{ard_1d_3} = 1$$

Therefore,

$$u_2^{rd_1d_3}=1 \text{ and } u_1^{rd_1d_2}=1.$$

Therefore $\Gamma/\Gamma' \cong z_{rd_1d_2} \oplus z_{rd_1d_3}$.

By application of Lemma (2.1.1) we have

$$\Gamma' = \left(\gamma';\ \underbrace{a,\dots,a}_{rd_1d_3-\text{times}},\ \underbrace{b,\dots,b}_{rd_1d_2-\text{times}},\ \underbrace{c,\dots,c}_{rd_1^2-\text{times}},\ \underbrace{\mu,\dots,\mu}_{r^2d_1^2d_2d_3-\text{times}}\right). \qquad \text{......(4.1.4.1)}$$

where γ' is calculated from (2.1.1) and is

$$\frac{1}{2}r[rd_1^2d_2d_3-d_1d_3-d_1d_2-d_1^2]+1. \qquad \text{......(4.1.4.2)}$$

Since Γ' satisfies the l.c.m. condition, therefore Γ'' is a surface group. By application of Lemma (2.1.3), let us define a subgroup,

$$\Gamma_k = \{x^{abc\mu k},\ x \in \Gamma' \text{ for any positive integer } k \geq 1\} \text{ of } \Gamma',$$

where $abc\mu$ is the l.c.m. of the periods of Γ'.

Let $N = \Gamma_k\Gamma''$ and N is normal in Γ'.

Further $\Gamma'' \subseteq \Gamma_k\Gamma'' = N$.

Therefore Γ'/N is abelian as it contains Γ''.

Considering an abelianizing homomorphism :

$$\phi : \Gamma' \to \Gamma'/N$$

we have

$$\Gamma'/N \cong \underbrace{z_a \oplus \dots \oplus z_a}_{(rd_1d_3-1)\text{ summands}} \oplus \underbrace{z_b \oplus \dots \oplus z_b}_{(rd_1d_2-1)\text{ summands}} \oplus \underbrace{z_c \oplus \dots \oplus z_c}_{(rd_1^2-1)\text{ summands}}$$

$$\oplus \underbrace{z_\mu \oplus \dots \oplus z_\mu}_{(r^2d_1^2d_2d_3-1)\text{ summands}} \oplus \underbrace{z_{abc\mu k} \oplus \dots \oplus z_{abc\mu k}}_{2\gamma'\text{ summands}}.$$

Therefore,

$$\left|\Gamma'/N\right| = a^{rd_1d_3-1}b^{rd_1d_2-1}c^{rd_1^2-1}\mu^{r^2d_1^2d_2d_3-1}(abc\mu k)^{2\gamma'}$$

$$= a^{rd_1d_3+2\gamma'-1}b^{rd_1d_2+2\gamma'-1}c^{rd_1^2+2\gamma'-1}\mu^{r^2d_1^2d_2d_3+2\gamma'-1}k^{2\gamma'}, \text{ for } k \geq 1$$

$$= A, \text{ say.} \qquad \text{......(4.1.4.3)}$$

Applying Lemma (2.1.1) we have N = $(\gamma''; \ldots)$ which shows that N contains no finite order generators. Thus N is a surface group and the genus γ'' is calculated from (2.1.1) is

$$\frac{A}{2}\left[2r^2d_1^2d_2d_3 - \frac{rd_1d_3}{a} - \frac{rd_1d_2}{b} - \frac{rd_1^2}{c} - \frac{r^2d_1^2d_2d_3}{\mu}\right] + 1 \qquad \text{......(4.1.4.4)}$$

where A is given by (4.1.4.3).

If we set $G = \Gamma/N$, then G is a smooth quotient of Γ as N is a normal surface group.

Also $G' = \Gamma'/N, G'' = \Gamma''/N = \{1\}$ as $\Gamma'' \subseteq N$.

Therefore $G \trianglerighteq G' \trianglerighteq G'' = \{1\}$ showing that G is metabelian.

Now $\quad |G| = \left|\Gamma/N\right|$.

$$= \left|\Gamma/\Gamma'\right|\left|\Gamma'/N\right|$$

$$= r^2d_1{}^2d_2d_3A$$

$$= Ar^2d_1{}^2d_2d_3 \qquad \text{......(4.1.4.5)}$$

where A is given by (4.1.4.3).

This completes the proof of the theorem.

The above theorem gives the following corollary.

Corollary 1 : If a = b = c = 1 in theorem (4.1.4) then Γ admits a metabelian smooth quotients of order

$$r^2d_1^2d_2d_3\mu^{r^2d_1^2d_2d_3+2\gamma'-1}k^{2\gamma'}, k \geq 1$$

and of genus

$$\frac{1}{2}\mu^{r^2d_1^2d_2d_3+2\gamma'-1}k^{2\gamma'}\left[\, 2r^2d_1^2d_2d_3 - rd_1d_3 - rd_1d_2 - rd_1^2 - \frac{r^2d_1^2d_2d_3}{\mu} \,\right]+1$$

where $\gamma' = \frac{1}{2}r[rd_1^2d_2d_3 - d_1d_3 - d_1d_2 - d_1^2]+1$.

Proof : From (4.1.4.1) we get

$$\Gamma' = \left(\gamma'; \ \underbrace{\mu,\ldots,\mu}_{r^2d_1^2d_2d_3-\text{times}} \right)$$

and $\quad \gamma' = \frac{1}{2}r[rd_1^2d_2d_3 - d_1d_3 - d_1d_2 - d_1^2]+1$,

which shows that Γ'' is a surface group as the periods of Γ' satisfies the l.c.m. condition.

By applying Lemma (2.1.3), let us define a subgroup

$\Gamma_k = \{x^{\mu k}, x \in \Gamma'$ for any positive integer $k \geq 1\}$ of Γ',

where μ is the l.c.m. of the periods of Γ'.

Let $N = \Gamma_k\Gamma''$ and N is normal in Γ'.

Further $\Gamma'' \subseteq \Gamma_k\Gamma'' = N$.

Therefore Γ'/N is abelian as it contains Γ''.

Considering an abelianizing homomorphism $\phi : \Gamma' \to \Gamma'/N$

we have $\quad \Gamma'/N \cong \underbrace{z_\mu \oplus \ldots \oplus z_\mu}_{(r^2d_1^2d_2d_3-1)\text{ summands}} \oplus \underbrace{z_{\mu k} \oplus \ldots \oplus z_{\mu k}}_{2\gamma'\text{ summands}}$.

Therefore $\quad \left|\Gamma'/N\right| = \mu^{(r^2d_1^2d_2d_3-1)}(\mu k)^{2\gamma'}$

$$= \mu^{r^2d_1^2d_2d_3+2\gamma'-1}k^{2\gamma'}, k \geq 1.$$

Applying Lemma (2.1.1) we have $N = (\gamma''; \ldots)$ which shows that N contains no finite order generators. Thus N is a surface group and the genus γ'' is calculated from (2.1.1) is

$$\frac{1}{2}\mu^{r^2d_1^2d_2d_3+2\gamma'-1}k^{2\gamma'}\left[2r^2d_1^2d_2d_3 - rd_1d_3 - rd_1d_2 - rd_1^2 - \frac{r^2d_1^2d_2d_3}{\mu}\right]+1$$

where $\gamma' = \frac{1}{2}r[rd_1^2d_2d_3 - d_1d_3 - d_1d_2 - d_1^2]+1$.

If we set $G = \Gamma/N$, then G is a smooth quotient of Γ and N is a normal surface group.

Also $G' = \Gamma'/N, G'' = \Gamma''/N = \{1\}$ as $\Gamma'' \subseteq N$.

Therefore $G \trianglerighteq G' \trianglerighteq G'' = \{1\}$ showing that G is metabelian.

Now $\quad |G| = \left|\Gamma/\Gamma'\right|\left|\Gamma'/N\right|$

$$= r^2d_1^2d_2d_3\mu^{r^2d_1^2d_2d_3+2\gamma'-1}k^{2\gamma'}, k \geq 1.$$

This completes the proof.

Theorem 4.1.5 : Let $\Gamma = (\ell, m, n, \mu)$ be a Fuchsian group where ℓ, m, n, μ are positive integers greater than or equal to 2 such that

$$(\ell, m) = d_1 > 1,\ (\ell, n) = d_2 > 1,\ (\ell, \mu) = d_3 > 1,\ (m, n) = d_4 > 1$$

and $\quad (m, \mu) = (n, \mu) = 1,$

where d_1, d_2, d_3, d_4 are pairwise prime to each other. Then Γ admits a metabelian smooth quotient of order

$$Ad_1d_2^2d_3d_4$$

and of genus

$$\frac{A}{2}\left[2d_1d_2^2d_3d_4 - \frac{d_2d_4}{a} - \frac{d_2^2d_3}{b} - \frac{d_1d_2d_3}{c} - \frac{d_1d_2^2d_4}{d}\right]+1,$$

where $A = a^{d_2d_4+2\gamma'-1} b^{d_2^2d_3+2\gamma'-1} c^{d_1d_2d_3+2\gamma'-1} d^{d_1d_2^2d_4+2\gamma'-1} k^{2\gamma'}, k \geq 1$

and $\gamma' = \frac{1}{2}[2d_1d_2^2d_3d_4 - d_2d_4 - d_2^2d_3 - d_1d_2d_3 - d_1d_2^2d_4] + 1$.

Proof : Let us take $\ell = ad_1d_2d_3$, $m = bd_1d_4$, $n = cd_2d_4$ and $\mu = dd_3$ where a, b, c, d are pairwise prime to each other.

Let Γ be generated by elements x_1, x_2, x_3 and x_4 satisfying :

$$x_1^{ad_1d_2d_3} = x_2^{bd_1d_4} = x_3^{cd_2d_4} = x_4^{dd_3} = x_1x_2x_3x_4 = 1$$

or equivalently,

$$x_1^{ad_1d_2d_3} = x_2^{bd_1d_4} = x_3^{cd_2d_4} = (x_1x_2x_3)^{dd_3} = 1.$$

Let u_1, u_2, u_3 and u_4 be the images of x_1, x_2, x_3 and x_4 respectively under the abelianizing homomorphism from Γ to Γ/Γ'. Then Γ/Γ' is generated by u_1, u_2, u_3 and u_4 satisfying :

$$u_1^{ad_1d_2d_3} = u_2^{bd_1d_4} = u_3^{cd_2d_4} = u_4^{dd_3} = u_1u_2u_3u_4 = 1$$

or equivalently,

$$u_1^{ad_1d_2d_3} = u_2^{bd_1d_4} = u_3^{cd_2d_4} = (u_1u_2u_3)^{dd_3} = 1.$$

From the above relations we have, $u_2 = \left(u_3u_1^{d_3}\right)^{-1}$

$$u_1^{d_1d_2d_3} = u_3^{d_2d_4} = 1.$$

Therefore $\Gamma/\Gamma' \cong z_{d_1d_2d_3} \oplus z_{d_2d_4}$

and $\left|\Gamma/\Gamma'\right| = d_1d_2^2d_3d_4$.

By application of Lemma (2.1.4) we have

$$\Gamma' = \left(\gamma';\ \underbrace{a,\ldots,a}_{d_2d_4\text{-times}},\ \underbrace{b,\ldots,b}_{d_2^2d_3\text{-times}},\ \underbrace{c,\ldots,c}_{d_1d_2d_3\text{-times}},\ \underbrace{d,\ldots,d}_{d_1d_2^2d_4\text{-times}} \right) \qquad \ldots\ldots(4.1.5.1)$$

where $\gamma' = \frac{1}{2}[2d_1d_2^2d_3d_4 - d_2d_4 - d_2^2d_3 - d_1d_2d_3 - d_1d_2^2d_4] + 1$.

Since Γ' satisfies the l.c.m. condition, therefore Γ'' is a surface group.

By application of Lemma (2.1.3), let us define a subgroup,

$$\Gamma_k = \{x^{pk},\ x \in \Gamma' \text{ for any positive integer } k \geq 1\} \text{ of } \Gamma',$$

where p is the l.c.m. of the periods of Γ' [$p = abcd$].

Let $N = \Gamma_k\Gamma''$ and N is normal in Γ'.

Further since $\Gamma'' \subseteq \Gamma_k\Gamma'' = N$, therefore Γ'/N is abelian as it contains Γ''.

Considering an abelianizing homomorphism $\phi : \Gamma' \rightarrow \Gamma'/N$ such that $u_1,\ldots,u_{d_2d_4}, v_1,\ldots,v_{d_2^2d_3}, w_1,\ldots,w_{d_1d_2d_3}, w'_1,\ldots,w'_{d_1d_2^2d_4}, a_1, b_1,\ldots,a_{\gamma'}, b_{\gamma'}$ be the images of $x'_1,\ldots,x'_{d_2d_4}, y'_1,\ldots,y'_{d_2^2d_3}, z'_1,\ldots,z'_{d_1d_2d_3}, z_1,\ldots,z_{d_1d_2^2d_4}, \alpha_1, \beta_1,\ldots,\alpha_{\gamma'}, \beta_{\gamma'}$ respectively satisfying the conditions :

$$u_1^a = \ldots = u_{d_2d_4}^a = v_1^b = \ldots = v_{d_2^2d_3}^b = w_1^c = \ldots = w_{d_1d_2d_3}^c = w_1'^d = \ldots = w'^d_{d_1d_2^2d_4}$$

$$= u_1 \ldots u_{d_2d_4} v_1 \ldots v_{d_2^2d_3} w_1 \ldots w_{d_1d_2d_3} w'_1 \ldots w'_{d_1d_2^2d_4} = 1$$

$$\text{and} \quad u_1^{pk} = \ldots = u_{d_2d_4}^{pk} = v_1^{pk} = \ldots = v_{d_2^2d_3}^{pk} = w_1^{pk} = \ldots = w_{d_1d_2d_3}^{pk} = w_1'^{pk} = \ldots = w'^{pk}_{d_1d_2^2d_4} = 1$$

also the elements commute with each other.

Therefore,

$$\Gamma'/N \cong \underbrace{Z_a \oplus \ldots \oplus Z_a}_{(d_2d_4-1)\text{ summands}} \oplus \underbrace{Z_b \oplus \ldots \oplus Z_b}_{(d_2^2d_3-1)\text{ summands}} \oplus \underbrace{Z_c \oplus \ldots \oplus Z_c}_{(d_1d_2d_3-1)\text{ summands}}$$

$$\oplus \underbrace{Z_d \oplus \ldots \oplus Z_d}_{(d_1d_2^2d_4-1)\text{ summands}} \oplus \underbrace{Z_{pk} \oplus \ldots \oplus Z_{pk}}_{2\gamma'\text{ summands}}$$

Therefore $\quad \left|\Gamma'/N\right| = a^{d_2d_4-1}b^{d_2^2d_3-1}c^{d_1d_2d_3-1}d^{d_1d_2^2d_4-1}(pk)^{2\gamma'}$

$$= a^{d_2d_4+2\gamma'-1}b^{d_2^2d_3+2\gamma'-1}c^{d_1d_2d_3+2\gamma'-1}d^{d_1d_2^2d_4+2\gamma'-1}k^{2\gamma'}, \text{ for } k \geq 1$$

$$= A, \text{ say.} \qquad \ldots\ldots(4.1.5.2)$$

By applying Lemma (2.1.1) we have N = (γ''; ...) which shows that N contains no finite order generators. Thus N is a surface group whose genus is

$$\frac{A}{2}\left[2d_1d_2^2d_3d_4 - \frac{d_2d_4}{a} - \frac{d_2^2d_3}{b} - \frac{d_1d_2d_3}{c} - \frac{d_1d_2^2d_4}{d}\right] + 1 \qquad \ldots\ldots(4.1.5.3)$$

where A is given by (4.1.5.2).

We now set $G = \Gamma/N$, then G is a smooth quotient of Γ and N is a normal surface group.

Also $G' = \Gamma'/N, G'' = \Gamma''/N = \{1\}$ as $\Gamma'' \subseteq N$.

Therefore $G \trianglerighteq G' \trianglerighteq G'' = \{1\}$ showing that G is metabelian.

Now $\quad |G| = \left|\Gamma/N\right|$.

$$= \left|\Gamma/\Gamma'\right|\left|\Gamma'/N\right|$$

$$= d_1d_1^2d_3d_4A$$

$$= Ad_1d_1^2d_3d_4 \qquad \ldots\ldots.(4.1.5.4)$$

where A is given by (4.1.5.2). This completes the proof of the theorem.

The above theorem gives the following corollary.

Corollary 1 : If a = b = c = d = 1, then Γ admits an abelian smooth quotients of order $d_1d_2{}^2d_3d_4$ and of genus $\frac{1}{2}[2d_1d_2^2d_3d_4 - d_2d_4 - d_2^2d_3 - d_1d_2d_3 - d_1d_2^2d_4] + 1$ as well as metabelian smooth quotient of order $k^{2\gamma'}d_1d_2^2d_3d_4$ and of genus

$$\frac{1}{2}k^{2\gamma'}[2d_1d_2^2d_3d_4 - d_2d_4 - d_2^2d_3 - d_1d_2d_3 - d_1d_2^2d_4] + 1, \text{ for } k \geq 1$$

where $\gamma' = \frac{1}{2}[2d_1d_2^2d_3d_4 - d_2d_4 - d_2^2d_3 - d_1d_2d_3 - d_1d_2^2d_4] + 1$.

Proof : From (4.1.5.1) we have $\Gamma' = (\gamma'; \ldots)$ which shows that Γ admits abelian smooth quotient of order $d_1d_2{}^2d_3d_4$ and of genus

$$\frac{1}{2}[2d_1d_2^2d_3d_4 - d_2d_4 - d_2^2d_3 - d_1d_2d_3 - d_1d_2^2d_4] + 1.$$

Now we apply Lemma (2.1.2) and define a subgroup

$\Gamma_k = \{x^k, x \in \Gamma' \text{ for } k \geq 1\}$ of Γ' containing Γ''.

Let $N = \Gamma_k\Gamma''$, then N is normal subgroup of Γ' of finite index.

As $\Gamma'' \subseteq \Gamma_k\Gamma'' = N$, therefore ${}^{\Gamma'}\!/_{N}$ is abelian.

Consider an abelianizing homomorphism :

$$\phi : \Gamma' \rightarrow {}^{\Gamma'}\!/_{N}$$

we have $${}^{\Gamma'}\!/_{N} \cong \underbrace{z_k \oplus \ldots \oplus z_k}_{2\gamma' \text{ summands}}.$$

Applying Lemma (2.1.1) we have $N = (\gamma''; \ldots)$ which shows that N contains no finite order generators. Thus N is a surface group. The genus γ'' is calculated from (2.1.1) is

$$\gamma'' = \frac{1}{2}k^{2\gamma'}[2d_1d_2^2d_3d_4 - d_2d_4 - d_2^2d_3 - d_1d_2d_3 - d_1d_2^2d_4] + 1.$$

If we set $G = {\Gamma}/{N}$, then G is a smooth quotient of Γ, since N is a surface group.

Also $G' = {\Gamma'}/{N}, G'' = {\Gamma''}/{N} = \{1\}$ as $\Gamma'' \subseteq N$.

Therefore $G \trianglerighteq G' \trianglerighteq G'' = \{1\}$. Hence G is metabelian.

Now the order of G is $|G| = \left|{\Gamma}/{N}\right| = \left|{\Gamma}/{\Gamma'}\right|\left|{\Gamma'}/{N}\right|$

$$= k^{2\gamma'}d_1d_2^2d_3d_4, \text{ for } k \geq 1$$

where $\gamma' = \frac{1}{2}[2d_1d_2^2d_3d_4 - d_2d_4 - d_2^2d_3 - d_1d_2d_3 - d_1d_2^2d_4] + 1.$

The following cases were studied by Parbin Ahmed [9] in her doctoral thesis which will comes out as special cases in our study.

I. If u, v, w are positive integers greater than 1 such that u < v < w and (u, v) = (u, w) = (v, w) = 1 then for every positive integer k, (u, u, v, w) has a solvable smooth quotient G_k of order $uv^{u-1}w^{u-1}k^{2g}$ and of derived length ≤ 3, where

$$2g = v^{u-2}w^{u-2}[2(u-1)vw - uv - uw] + 2,$$

acting as an automorphism group on a surface of genus η given by

$$2(\eta - 1) = v^{u-2}w^{u-2}k^{2g}[2uvw - 2vw - uv - uw].$$

II. If u, v, w are positive integers >1 such that u < v < w and (u, v) = (u, w) = (v, w) = 1 then for every positive integer k, (u, v, w, uvw) has a solvable smooth quotient G_k of order $uvwk^{2g}$ and of derived length ≤ 2 where

$$2g = 2uvw - (uv + uw + vw) + 1$$

acting as an automorphism group of a surface of genus η given by

$$2(\eta - 1) = k^{2g}[2uvw - (uv + uw + vw) - 1].$$

III. If u, v, w, z are odd positive integers and z > 1, and (u, v) = (u, w) = (u, z) =

(v, w) = (v, z) = (w, z) = 1 then for every positive integer k, (2u, 2v, 2w, z) has a solvable smooth quotient G_k of order $4uvwz^3k^{2g}$ and of derived length ≤ 3, where

$$2g = z^2[8uvwz - 2(vwz + uwz + uvz - 4uvw)] + 2$$

acting as an auotomorphism group of a surface of genus η given by

$$2(\eta - 1) = 2z^2[4uvwz - (vwz + uwz + uvz) - 2uvw]k^{2g}.$$

IV. If u is a positive integer greater than 2, then for every positive integer k, (u, u, u, u) has a solvable smooth quotient G_k of order u^3k^{2g} where $g = u^3 - 2u^2 + 1$ which acts as an automorphism group of a surface of genus η given by

$$2(\eta - 1) = u^2k^{2g}[2u - 4].$$

V. If u, v are positive integers greater than 1 then for every positive integer k, (u, u, uv, uv) has a solvable smooth quotient G_k of order u^3vk^{2g} where

$$g = u^2[uv - v - 1] + 1$$

acting as an automorphism group of a surface of genus η given by

$$2(\eta - 1) = u^2k^{2g}[2uv - 2 - 2v].$$

VI. If u, v, w are positive integers > 1 such that (v, w) = 1 then for every positive integer k, (u, uv, uw, uvw) has a solvable smooth quotient G_k of order u^3vwk^{2g} where

$$2g = u^2[vw(2u - 1) - (v + w + 1)] + 2$$

which acts as an automorphism group of a surface of η given by

$$2(\eta - 1) = u^2k^{2g}[2uvw - (vw + v + w) - 1].$$

4.2 Quadruple extension of quadruple groups

A group G is an extension of a group H by another group K if G contains a normal subgroup $\overline{H}$ isomorphic to H such that $G/\overline{H} \cong K$. The simplest way to construct an extension of H by K is to form the direct product G = H × K for, it is known that G contains two normal subgroups $\overline{H} = \{(h,1) \mid h \in H\}$ and $\overline{K} = \{(1,k) \mid k \in K\}$ with the properties $\overline{H} \cong H, \overline{K} \cong K, G/\overline{H} \cong K$ and $G/\overline{K} \cong H$.

We now prove below a theorem 4.2.1 which will prove that an extension of a quadruple group by a quadruple group is again a quadruple group.

Theorem 4.2.1 : Let H be a quadruple group of type $(\lambda, \mu, \nu, \eta)$ and K be a simple quadruple group of type (ℓ, m, n, ξ) such that at least one of the conditions :

$\ell \nmid \lambda$,

$m \nmid \mu$,

$n \nmid \nu$,

$\xi \nmid \eta$ holds, then $G = H \times K$ is a quadruple group of type

$([\lambda, \ell], [\mu, m], [\nu, n], [\eta, \xi])$.

Proof : Let H be generated by u, v, w satisfying $u^\lambda = v^\mu = w^\nu = (uvw)^\eta = 1$ and K be generated by x, y, z satisfying

$x^\ell = y^m = z^n = (xyz)^\xi = 1$.

Now $G = H \times K$ can be generated by (u, 1), (v, 1), (w, 1), (1, x), (1, y) and (1, z).

Let $\wedge$ be the subgroup of G generated by (u, x), (v, y) and (w, z). We claim that $\wedge = G$. We observe that $\overline{K} \cong K$ and is generated by (1, x), (1, y) and (1, z). Now consider the subgroup $D = \wedge \cap \overline{K}$ of G. We are given that at least one of the conditions $\ell \nmid \lambda$, $m \nmid \mu$, $n \nmid \nu$ and $\xi \nmid \eta$ holds :

When $\ell \nmid \lambda$ we have

$$(u, x)^\lambda = (1, x^\lambda) = (1, x)^\lambda \in \overline{K}$$

so that $(1, x^\lambda) \in \wedge \cap \overline{K}$

and $(1, x^\lambda) \neq (1, 1)$.

Hence D is not the identity subgroup.

When $m \nmid \mu$ we have

$$(v, y)^\mu = (1, y^\mu) = (1, y)^\mu \in \overline{K}$$

so that $(1, y^\mu) \in \wedge \cap \overline{K}$

and $(1, y^\mu) \neq (1, 1)$.

Hence D is not the identity subgroup. Similarly

When n $\not|$ ν we have

$$(w, z)^{\nu} = (1, z^{\nu}) = (1, z)^{\nu} \in \overline{K}$$

so that $(1, z^{\nu}) \in \wedge \cap \overline{K}$

and $(1, z^{\nu}) \neq (1, 1)$.

Hence D is not the identity subgroup.

Lastly when $\xi \not| \eta$ we have

$$\begin{aligned}\{(u, x)(v, y)(w, z)\}^{\eta} = (uvw, xyz)^{\eta} &= (1, (xyz)^{\eta}) \\ &= (1, (xyz))^{\eta} \\ &= \{(1, x)(1, y)(1, z)\}^{\eta} \\ &\in \overline{K}\end{aligned}$$

and $(1,(xyz)^{\eta}) \neq (1, 1)$.

Hence in this case also D is not the identity subgroup. Now D is a subgroup of $\wedge$ as well as $\overline{K}$. Since $\overline{K}$ is normal in G, it follows that $D = \wedge \cap \overline{K}$ is a normal subgroup of $\wedge$, since $\wedge$ is also a subgroup of G.

Let us now define

$$\Pi : \wedge \rightarrow \overline{K} \text{ by } \Pi(\xi_1, \xi_2) = (1, \xi_2).$$

Then Π is a homomorphism. Further Π maps (u, x), (v, y), (w, z) onto the generators (1, x), (1, y), (1, z) respectively of $\overline{K}$. Hence Π is an epimorphism. Also Π maps D onto itself. Since D is normal in G and non-trivial, D is a non-trivial normal subgroup of $\overline{K}$, but K is simple and therefore $\overline{K}$, being an isomorphic image of a simple group, is simple. This shows that $\overline{K}$ coincides with D. Hence

$$\wedge \supseteq \overline{K}\cdot$$

Therefore

$$(1, x), (1, y), (1, z) \in \wedge.$$

Now $(u, 1) = (u, x)(1, x^{\ell - 1}) \in \wedge$,

$(v, 1) = (v, y)(1, y^{m - 1}) \in \wedge$,

and $(w, 1) = (w, z)(1, z^{n - 1}) \in \wedge$.

Hence $(1, x), (1, y), (1, z), (u, 1), (v, 1), (w, 1) \in \wedge$.

Therefore $G = \wedge$.

Now set $\alpha = (u, x)$, $\beta = (v, y)$, $\gamma = (w, z)$.

Then G is generated by α, β, γ satisfying :

$$\alpha^{[\lambda, \ell]} = \beta^{[\mu, m]} = \gamma^{[\nu, n]} = (\alpha\beta\gamma)^{[\eta, \xi]} = 1.$$

Hence G is a quadruple group of type :

$$([\lambda, \ell], [\mu, m], [\gamma, n], [\eta, \xi]).$$

Theorem 4.2.2 : If $G = H \times K$ is an extension of H by K then $G' = H' \times K'$ is also an extension of H' by K', where G', H', K' are the derived groups of G, H and K respectively.

Proof : We first show that H' and K' are two normal subgroups of G' then $G' = H' \times K'$. We now show that H' is normal in G'.

Let $x \in G'$ therefore $x = a_1b_1a_1^{-1}b_1^{-1}$ for some $a_1, b_1 \in G$.

Let $h \in H'$ then $h \in H$ [since $H' \underline{\Delta} H$].

Since $a_1 \in G$ and $h \in H$ and $H \underline{\Delta} G$ therefore $a_1ha_1^{-1} \in H$.

Now to show that $xhx^{-1} \in H'$.

$$\begin{aligned} xhx^{-1} &= (a_1b_1a_1^{-1}b_1^{-1})h(b_1a_1b_1^{-1}a_1^{-1}) \\ &= a_1b_1a_1^{-1}(b_1^{-1}hb_1)a_1b_1^{-1}a_1^{-1}. \\ &= a_1b_1(a_1^{-1}h_1a_1)b_1^{-1}a_1^{-1} && [b_1^{-1}hb_1 \in H' = h_1, \text{ say}] \\ &= a_1(b_1h_2b_1^{-1})a_1^{-1} && [a_1^{-1}h_1a_1 = h_2, \text{ say}] \\ &= a_1h_3a_1^{-1} && [b_1h_2b_1^{-1} = h_3, \text{ say}] \\ &\in H'. \end{aligned}$$

Therefore H' is normal in G'.

Similarly we can show that K' is also normal subgroup of G'. Therefore G' contains two normal subgroups H' and K'. Next to show that

$$G' = H' \times K'$$

First to show that $G' \subseteq H' \times K'$.

Let $x \in G'$, therefore $x = [a, b]$ where $a, b \in G$.

We have $G = H \times K$ an extension of H by K.

Since $a, b \in G$, therefore $a = (h_1, k_1)$, $b = (h_2, k_2)$ where $h_1, h_2 \in H$ and $k_1, k_2 \in K$.

Now $x = [a, b] = [(h_1, k_1), (h_2, k_2]$

$$= (h_1^{-1}, k_1^{-1})(h_2^{-1}, k_2^{-1})(h_1, k_1)(h_2, k_2)$$

$$= (h_1^{-1}h_2^{-1}h_1h_2, k_1^{-1}k_2^{-1}k_1k_2)$$

$$= ([h_1, h_2], [k_1, k_2])$$

$$\in H' \times K'.$$

Therefore $x \in G' \Rightarrow x \in H' \times K'$.

Hence $G' \subseteq H' \times K'$ ......(4.2.2.1)

Conversely, let

$x \in H' \times K'$

i.e. $x = (h, k) \in H' \times K'$ where $h \in H'$, $k \in K'$.

Let $h = [a_1, b_1]$ and $k = [a_2, b_2]$ for some $a_1, b_1 \in H$ and $a_2, b_2 \in K$.

Now $x = (h, k) = ([a_1, b_1], [a_2, b_2])$

$$= (a_1b_1a_1^{-1}b_1^{-1}, a_2b_2a_2^{-1}b_2^{-1})$$

$$= (a_1, a_2)(b_1, b_2)(a_1^{-1}, a_2^{-1})(b_1^{-1}, b_2^{-1})$$

$$= (a_1, a_2)(b_1, b_2)\{(a_1, a_2)\}^{-1}\{(b_1, b_2)\}^{-1}$$

$$= [(a_1, a_2), (b_1, b_2)]$$

$$\in G'.$$

Therefore, $x \in H' \times K' \Rightarrow x \in G'$

Hence $H' \times K' \subseteq G'$ ......(4.2.2.2)

From (4.2.2.1) and (4.2.2.2) we get,

$$G' = H' \times K'$$

Therefore if $G = H \times K$ is an extension of H by K then $G' = H' \times K'$ is also an extension of H' by K'.

This completes the proof.

The construction of the extensions of admissible quadruple groups will be followed after the following definition :

Definition 4.2.1 : Let G be a finite group with a sequence of subgroups $G = G^{(0)} \supseteq G^{(1)} \supseteq \ldots$ where each group $G^{(i)}$ is the derived group of $G^{(i-1)}$ for $i = 1, 2, \ldots$; which terminates in the identity in a finite number of steps, say $G^{(n)} = 1$, then G is called ***solvable*** of ***length n***. Now each factor group G^{i-1}/G^{i} is abelian. If $G^{(i)} = G^{i+1}$, then we have

$$G^{(i)} = G^{(j)} \text{ for all } j \geq i.$$

If there exists a least non-negative integer n satisfying $G^{n} = G^{n+1} = \ldots$ then we say G is insolvable of length n. Solvable groups of length 1 and 2 are abelian and metabelian groups respectively where as insolvable groups of length O are the perfect groups. A finite group G generated by three elements x, y and z satisfying the relations :

$$x^{\ell} = y^{m} = z^{n} = (xyz)^{\xi} = 1$$

is called a ***quadruple group***. It is to be noted that the cyclic group z_m can be viewed as a quadruple group of type (m, m, m, 1). Similarly $z_{\ell} \oplus z_m$ can also be thought of as a quadruple group of type (ℓ, m, [ℓ, m], 1) where [ℓ, m] denotes the l.c.m. of ℓ and m.

Definition 4.2.2 : Let H be a quadruple group of type (λ, μ, ν, η) with the generators u, v and w satisfying the relations :

$$u^{\lambda} = v^{\mu} = w^{\nu} = (uvw)^{\eta} = 1.$$

Let p be a prime such that p does not divide μ and ν and η. Now the group z_p i.e. (p, p, p, 1) is simple and therefore by theorem 4.2.1 the group $G = H \times z_p$ is a quadruple extension of H by z_p i.e.

$$H \times z_p = \big([\lambda, p], [\mu, p], [\nu, p], \eta\big).$$

We know that an extension of a solvable group by a cyclic group is solvable.

Let us now discuss a method to construct a series of admissible quadruple groups G_i, $i = 1, \ldots, k$ such that given any positive integer k and an admissible quadruple group G,

$$G \subset G_1 \subset \ldots \subset G_k$$

and $\quad G' = G_i', \ 1 \leq i \leq k$

where G_i' is the derived group of G_i.

Theorem 4.2.3 : Let G be a quadruple group of type (ℓ, m, n, μ) where $(\ell, m) = d_1 > 1$ and $(\ell, n) = (\ell, \mu) = (m, n) = (n, \mu) = (m, \mu) = 1$ generated by u, v, w satisfying the relations :

$$u^\ell = v^m = w^n = (uvw)^\mu = 1.$$

Let $P = \{p_1, \ldots, p_n\}$ be a set of distinct primes such that $p_1 < p_2 < \ldots < p_n$; $p_i \geq 3$ [If, say $p_1 = 2$ then we get z_2 as a smooth quotient of (2, 2, 2, 2, 2, 2) which is not a quadruple Fuchsian group].

Then there exists a series of quadruple groups $G_1, \ldots, G_n$ where

$$G_n \supset G_{n-1} \supset \ldots \supset G_1 \supset G$$

and (a) the order of $G_i = p_1 p_2 \ldots p_i \mid G \mid$, $i = 1, 2, \ldots, n$.

(b) G_i' is the derived group of G_i, $i = 1, 2, \ldots, n$

$$\left| {}^{G_1}\!/_{G} \right| = p_1 \text{ and } \left| {}^{G_i}\!/_{G_{i-1}} \right| = p_i, \; i = 2, 3, \ldots, n.$$

Proof : Let G be a quadruple group of type (ℓ, m, n, μ).

Let $P = \{p_1, p_2, \ldots, p_n\}$ be a set of all distinct primes such that $p_1 < p_2 < \ldots < p_n$.

Let z_{p_1} i.e., $(p_1, p_1, p_1, 1)$ be a simple quadruple group satisfying the condition :

$$u_1^{p_1} = u_2^{p_1} = u_3^{p_1} = u_1 u_2 u_3 = 1.$$

Let G_1 be an extension of G by z_{p_1} i.e.

$$G_1 = G \times z_{p_1} = ([\ell, p_1], [m, p_1], [n, p_1], \mu)$$

$$= (\alpha_1, \alpha_2, \alpha_3, \mu) \text{ say,}$$

where $[\ell, p_1] = \alpha_1$, $[m, p_1] = \alpha_2$, $[n, p_1] = \alpha_3$, p_1 may be exactly equal to ℓ or m or n.

Now $|G_1| = |G| p_1$,

and $G_1' \cong G' \times z_{p_1}' = G'$.

Therefore $G_1' \cong G'$ and $G_1 \supset G$ [since $G_1 \trianglelefteq G_1' \cong G'$].

Let p_2 be a prime, then the simple group will be of the form $(p_2, p_2, p_2, 1)$ i.e. z_{p_2}.

Let G_2 be an extension of G_1 by z_{p_2} i.e.

$$G_2 = G_1 \times z_{p_2} = ([\alpha_1, p_2], [\alpha_2, p_2], [\alpha_3, p_2], \mu)$$
$$= (\beta_1, \beta_2, \beta_3, \mu), \text{ say}$$

where $[\alpha_1, p_2] = \beta_1$, $[\alpha_2, p_2] = \beta_2$, $[\alpha_3, p_2] = \beta_3$.

Therefore $|G_2| = |G_1 \times z_{p_2}| = |G_1|\,|z_{p_2}| = p_1p_2\,|G|$

and $G_2' \cong G_1' \times z_{p_2}' = G_1' \cong G'$.

Therefore $G_2' \cong G'$ and $G_2 \supset G_1 \supset G$.

Continuing the process till the nth step we get a series of quadruple groups, $G_1, G_2, \ldots, G_n$ such that

$$G_n \supset G_{n-1} \supset \ldots \supset G_2 \supset G_1 \supset G$$

where $G_1 = G \times z_{p_1}$ and $G_i = G_{i-1} \times z_{p_i}$, $i = 2, 3, \ldots, n$ and $|G_i| = p_1p_2p_3 \ldots p_i|G|$, $i = 1, 2, 3, \ldots, n$.

Which completes the proof.

From the above theorem we can have a definition which is given below :

Definition 4.2.3 : Let G be a quadruple group of type (ℓ, m, n, μ) generated by u, v and w satisfying the relation :

$$u^\ell = v^m = w^n = (uvw)^\mu = 1.$$

Let $P = \{p_1, \ldots, p_n\}$ be a set of n-distinct primes such that $p_1 < p_2 < \ldots < p_n$, $p_i \geq 3$ then there exists a series of quadruple groups $G_1, \ldots, G_n$ such that

$$G \subset G_1 \subset \ldots \subset G_n$$

where $G_1 = G \times z_{p_1}$ and $G_i = G_{i-1} \times z_{p_i}$, $i = 2, 3, \ldots, n$. We define G_i to be the p_i-th extension of G.

In the above theorem, the choice of the primes $p_1, \ldots, p_n$ are arbitrary, the construction may be done in many ways. Moreover the simple cyclic group z_{p_i} might also be replaced by any known non-abelian simple quadruple groups. But in that case we may not have G as the derived group of G_i, we get a series of subgroups, which is insolvable.

4.3 Quadruple extension and Riemann surface automorphism groups

By our definition a quadruple group is a finite group. Also every finite group is an automorphism group of some compact Riemann surfaces and by theorem 4.2.1 quadruple extension is also a quadruple group. Hence an extension of quadruple groups acts as an automorphism groups of a compact Riemann surfaces.

Let G be a finite group. Now the genus g of the compact Riemann surface on which G acts is obtained from (2.1.1) as follows :

$$2(g-1)=|G|\left\{2-\frac{1}{\ell}-\frac{1}{m}-\frac{1}{n}-\frac{1}{\xi}\right\}$$

i.e., $$g=1+\frac{1}{2}|G|\left\{2-\frac{1}{\ell}-\frac{1}{m}-\frac{1}{n}-\frac{1}{\xi}\right\},$$

where (ℓ, m, n, ξ) is an admissible quadruple group satisfying :

$$\frac{1}{\ell}+\frac{1}{m}+\frac{1}{n}+\frac{1}{\xi}<2.$$

The above method may as well be applied to determine the genus of a compact Riemann surface admitting a p-extension of an admissible quadruple group as a group of automorphisms because of the following lemma.

Lemma 4.3.1 : A p-extension of an admissible quadruple group is also an admissible quadruple group.

Proof : Let G be an admissible quadruple group of type (ℓ, m, n, ξ). Then G is generated by u, v, w such that, $O(u) = \ell$, $O(v) = m$, $O(w) = n$ and $O(uvw) = \xi$ with

$$\frac{1}{\ell}+\frac{1}{m}+\frac{1}{n}+\frac{1}{\xi}<2.$$

Let p be a prime such that $p \nmid m$ and $p \nmid n$ and $p \nmid \xi$. Then by definition 4.2.2 $H = G \times z_p$ is a p-extension of G by z_p.

Let z_p be generated by w_1 so that $0(w_1) = p$. We see that H is generated by $\alpha = (u, 1)$, $\beta = (v, 1)$ and $\gamma = (w_1, y)$. Since ℓ, m, n, ξ and p are the respective orders of u, v, w, uvw and w_1 it follows that ℓ, [m, p], [n, p], [ξ, p] are the respective orders of α, β, γ and $\alpha\beta\gamma$. Further

$$\frac{1}{\ell}+\frac{1}{m}+\frac{1}{n}+\frac{1}{\xi}<2 \text{ implies}$$

$$\frac{1}{\ell}+\frac{1}{[m,p]}+\frac{1}{[n,p]}+\frac{1}{[\xi,p]}<2.$$

Therefore H is an admissible quadruple group. Which completes the proof.

Now we find the genus of the p_i-th extension G_i of G, which is stated in the theorem 4.2.3 .

Theorem 4.3.1 : Let G be an admissible quadruple group of type (ℓ, m, n, ξ) generated by u, v, w such that $p_1 < \dots < p_k$ and p_i not divide m_{i-1}, n_{i-1} and ξ_{i-1} where m_i, n_i and ξ_i are defined as follows :

(i) $m_0 = m$, $m_1 = [m_0, p_1]$, $m_2 = [m_1, p_2]$, ...

(ii) $n_0 = n$, $n_1 = [n_0, p_1]$, $n_2 = [n_1, p_2]$, ...

(iii) $\xi_0 = \xi$, $\xi_1 = [\xi_0, p_1]$, $\xi_2 = [\xi_1, p_2]$, ...

Then p_i-th extension G_i of G is an automorphism group of a compact Riemann surface of genus g_i, $1 \le i \le k$ where

$$g_i = 1+\frac{1}{2}(p_1 \dots p_i)|G|\left\{2-\frac{1}{\ell}-\frac{1}{m_i}-\frac{1}{n_i}-\frac{1}{\xi_i}\right\}.$$

Proof : From theorem 4.2.3 and Lemma 4.3.1 we observe that G_1, the p_1-extension of G is an admissible quadruple group. Also each G_i, $2 \leq i < k$ is the p_i-extension of G_{i-1}. By induction it now follows that G_i, the p_i-th extension of G, is an admissible quadruple group. Also from theorem 4.2.3 we see that,

$$|G_i| = (p_1 \ldots p_i)\ |G|$$

and we already have,

$$\frac{1}{\ell} + \frac{1}{m_i} + \frac{1}{n_i} + \frac{1}{\xi_i} < 2.$$

Therefore G_i is the automorphism group of a compact Riemann surface of genus g_i which satisfy the following equation :

$$\frac{2(g_i - 1)}{|G_i|} = 2 - \frac{1}{\ell} - \frac{1}{m_i} - \frac{1}{n_i} - \frac{1}{\xi_i}$$

i.e., $$g_i = 1 + \frac{1}{2}|G_i|\left\{2 - \frac{1}{\ell} - \frac{1}{m_i} - \frac{1}{n_i} - \frac{1}{\xi_i}\right\}$$

i.e., $$g_i = 1 + \frac{1}{2}(p_1 p_2 \ldots p_i)|G|\left\{2 - \frac{1}{\ell} - \frac{1}{m_i} - \frac{1}{n_i} - \frac{1}{\xi_i}\right\}.$$

This completes the proof.

4.4 Conclusion

From the theorems mentioned in section 4.1 of this chapter we have come to the conclusion that for any admissible quadruple group we can prove the existence of solvable smooth quotients whose derived length is less than or equal to 3.

○ ○

APPENDIX

SYMBOLS

The following is a list of symbols used in this thesis : —

$A = B$	The sets A and B are equal
$A \neq B$	The sets A and B are not equal
$A \subseteq B$	The set A is a subset of the set B
$A \subset B$	The set A is a proper subset of the set B
$A \supseteq B$	The set B is a subset of the set A
$A \supset B$	The set B is a proper subset of the set A
$A \cup B$	Union of two sets
$A \cap B$	Intersection of two sets
$A \times B$	Cartesian product of two sets
$a \equiv b(\mathrm{mod}\ m)$	Congruence modulo m for integers
$[\alpha, \beta]$	The commutator $\alpha^{-1}\beta^{-1}\alpha\beta$ of the elements α and β of a group
C	Complex number system
D	The upper half plane
D_{2p}	The dihedral group of order 2p, p is odd prime
D_{2n}	The dihedral group of order 2n

$\delta(\Gamma)$	$= 2(\gamma - 1) + \sum_{i=1}^{r}\left(1 - \frac{1}{m_i}\right)$ for a Fuchsian group Γ with signature $(\gamma; m_1, \dots, m_r)$
ϕ	The empty set
$f : X \to Y$	Function (or mapping) with domian X and range in Y
$f^{-1} : Y \to X$	Inverse function (or mapping)
$f(A)$	Image of a set under a mapping
$f^{-1}(B)$	Inverse image of a set under a mapping
$g.f : X \to Y$	Product of two mappings $f : X \to Y$ and $g : Y \to X$
$\|G\|$	Order of the group
$[G : H]$	Index of the subgroup H in the group G
$G \cong H$	G is isomorphic to H
$H \underline{\Delta} G$	H is a normal subgroup of G
$G^{(1)}$	The derived group of the group G
$G^{(i)}$	The ith derived group of the group G
G/N	The quotient group of G by the normal subgroup N
$(\gamma; m_1, \dots, m_r)$	Signature of a Fuchsian group with genus γ and $m_1, \dots, m_r$ as periods
$(\gamma; -)$	Signature of a surface group with genus γ
$H \oplus K$	Direct sum of two groups
(k_1, k_2)	The h. c. f of the integers k_1, k_2

$\{k_1, \ldots, k_r\}$	The l. c. m. of the integers $k_1, \ldots, k_r$
LF(2, $\mathbb{R}$)	The group of linear fractional transformations
$\ell \mid m$	The integer ℓ is a divisor of the integer m
$\ell \nmid m$	The integer ℓ is not a divisor of the integer m
$m < n$, $m \leq n$	Order relation for cardinal numbers
$\mu(E)$	Non-Euclidean measure of a measurable set E
$\mathbb{R}$	The set of real numbers
$\mathcal{K}$	A class of compact Riemann surfaces of algebraic genuses $g \geq 2$
$x \in A$	x belongs to the set A
$x \notin A$	x does not belong to the set A

* * *

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